Modelling stock returns with Lévy processes

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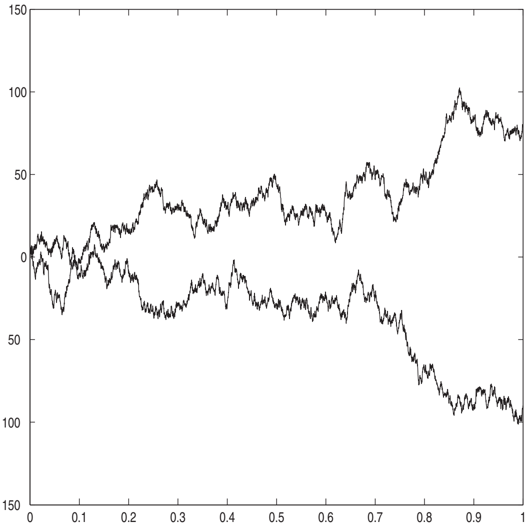
I Introduction

This article deals with the modelling of stock prices through non-stable Lévy processes. Before we introduce the central aim and content, let us first recall some historical facts about Brownian motion, which is a central tool in contemporary finance just as in all other domains of sciences. The name "Brownian" comes from Richard Brown, a botanist who studied pollens evolving in solutions in the early nineteenth century. In the twentieth century, physicists (Einstein, Perrin, Smoluchowsky) and mathematicians like Paul Lévy succeeded in giving a modelling of Brownian motion.

Figure 1. **Paths of standard Brownian motions**



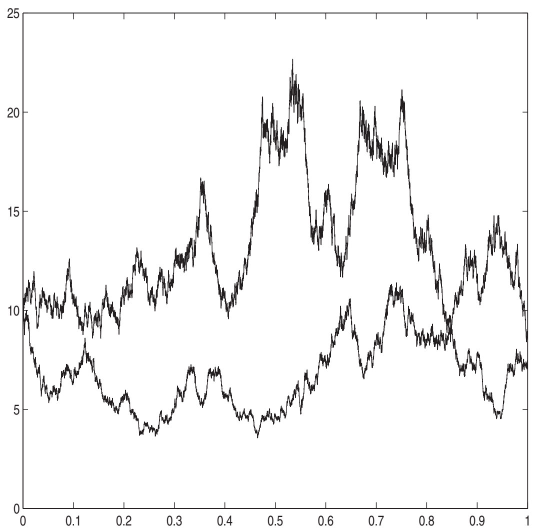
Amongst the properties of Brownian motion, one can cite independency and stationarity of increments, the latter property being often referred to as spatial homogeneity. These are indeed the defining properties of a Lévy process. In other words, a Brownian motion is a particular case of Lévy process. Arithmetic Brownian motions (general Brownian motions with drift) were first used in Finance to model stock price evolutions by Louis Bachelier in his state thesis in the year 1900. Some paths of Brownian motions are plotted in *figure 1*. If we take a look at these paths, we see that they visit the negative real line as often as they visit the positive one : modelling prices by Brownian motions would imply negative values for stock prices. This is not a good direct choice, but this does not prevent us to use Brownian motions as *indirect* tools to model stock prices.

In 1965 Samuelson, in the continuation of Osborne's ideas, suggested to use geometric Brownian motions instead of arithmetic ones. This can be considered an important step in mathematical finance : geometric Brownian motions provide a reasonably good fit to stock prices, they have the property of not visiting the negative real line and they are the core of the celebrated Black and Scholes model. This model, even in the numerous cases when it is insufficient, can be viewed nevertheless as a kind of reference : a particular model is often built according to the remedies it brings to the flaws of the Black and Scholes model. For example one can add stochastic volatility to account for the high/low activity periods observed in financial markets...

Hence this model and its central assumption – geometric Brownian motions to model stock price evolutions – are at the core of modern finance. We have plotted typical paths of geometric Brownian motions in *figure 2*.

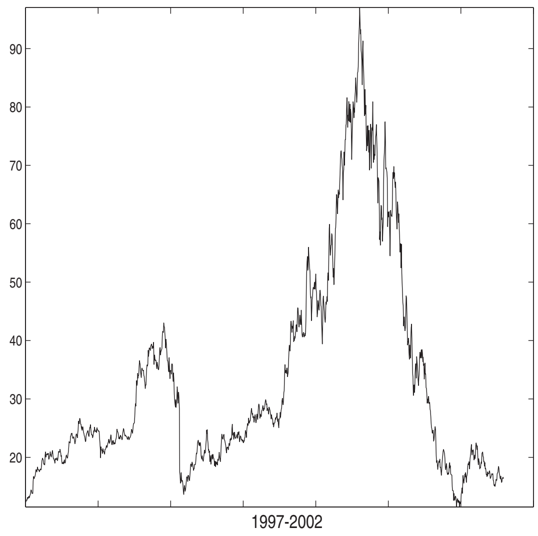
Special thanks are addressed to Frank Burjan, Jean-Paul Laurent, Paul Mc Gill, Yannick Malevergne, François Quittard-Pinon, Rivo Randrianarivony and the anonymous referee for their useful comments.

Figure 2. **Paths of geometric Brownian motions**



The first thing that can be observed is that this kind of stochastic process never jumps. It sometimes displays large changes, but it always keeps continuous paths. This is in contrast to the structure of everyday financial dynamics : consider for example the big jumps existing in Alcatel's recent years stock dynamics (*figure 3*).

Figure 3. **Evolution of Alcatel's stock value**



There is a need to incorporate jumps in stock price dynamics if one wishes to model them more precisely. The question that comes up then is : *what kind of processes should be used as substitutes to arithmetic or geometric Brownian motions in order to model asset evolutions?* The question at this stage is very general and requires some further developments and comments.

Firstly, there are many ways of building processes with jumps. Secondly, there are many other effects other than jumps that need to be modelled (dependence, changes in activity...). Thirdly, the evolution of a Libor rate will not be the same as the evolution of a bond or stock price. Consequently, the tools to model them will not be identical.

Thus we see the problem here : it is not possible to find a process generic enough to model all kind of assets and precise enough to deal with many various particular effects. There must be a trade-off between generality and specificity. Since geometric Brownian motions are extremely basic blocks of the family of stochastic processes, we believe that we can go into further generality and that some classes of Lévy processes (the family of processes with stationary and independent increments that in particular includes Brownian motion) – or their exponentials that mimic geometric Brownian motions – are good determinants for this purpose. They display jumps, are tractable and can be specified or used in the construction of more complicated processes, depending on the particular kind of application one is studying.

In 1976, Merton suggested to introduce compound Poisson processes (which are simple Lévy processes) in stock price dynamics in order to account for the presence of jumps. This could be seen as an attempt to improve the Black and Scholes framework which itself relies on Samuelson's ideas. Nevertheless, a few years before, Mandelbrot (see Mandelbrot (1997) for a global viewpoint on his models) had suggested to depart from the Gaussian paradigm as applied to return distributions and proposed to model returns by stable Lévy processes. The drawbacks of such an approach are widely well known : nearly all moments, and in particular moments of the second order, are not defined and there is a scale invariance in the process – due to the stable assumption. Under this scale invariance feature, a very particular vision of phenomena is lurking : according to this vision, same or analogous physical mechanisms exist at all time scales (traders should in a way behave if not identically at least similarly on a one minute, one week, one year basis) and the mathematical machinery should also be the same for all time scales. In this article we will not develop further on this approach, which has been well documented in the past forty years.

It is only in the past ten years that general *non-stable* Lévy processes started being deeply studied in representations of stock price dynamics. This article, which aims at describing such representations, is made of two distinct parts. The first part – consisting in sections two, three and four – makes a comparative review of all existing approaches. To our knowledge, there is at present no such review existing in the literature. The second part – made of the last section – is new. In this second part, we introduce non-stable Lévy processes that permit to take into account humped distributions. To us, the most interesting aspect of this model is that it allows for distributions varying in shape at all time scales, in deep contrast to the scale invariant approach of Mandelbrot. Let us now explain in more details the content of the sections of this article.

We will devote the first section to the exhibition of common facts on Lévy processes and on their use in the modelling of log-price dynamics. Secondly, we will present a class of distribution-parameterized approaches including the "general hyperbolic family", *idem est* a class of Lévy processes with general hyperbolic distributions. These processes have been extensively studied in Barndorff-Nielsen (1997, 1998), Eberlein *et alii* (1998), Raible (2000) and others. In a third section we will concentrate on another class of Lévy processes, namely the Variance Gamma and CGMY processes, and will group them under the name "Lévy density based approach". The Variance Gamma processes have been studied by Madan *et alii* (1990, 1991, 1998). Carr *et alii* (2002) extended these processes and built the CGMY model (from the names of the authors). In the final section, we will remain in a Lévy density based framework and introduce new processes that are titled $\alpha - \beta$ Lévy processes and are constructed so as to account for an excessive presence of average size jumps.

II The Lévy framework

After a general presentation on Lévy processes and their properties, we explain how they can be used in the modelling of stock price dynamics.

1. A brief account on Lévy processes

We recall basic facts on Lévy processes. They form the class of processes, owing càdlàg modifications, whose increments verify stationarity and independence. These processes share the Markov property and they are basic examples of semimartingales. Brownian motion – the fundamental object of finance – and Poisson processes – the fundamental objects of insurance – are the simplest yet most famous examples of Lévy processes.

Lévy processes are such as they can be described by only one distribution of an arbitrary underlying random variable. The knowledge of the law of, say, X_1 is sufficient to determine all the finite dimensional distributions $\{X_{t1}, X_{t2}, \ldots, X_{tn}\}$, hence the process. In the particular case of a Brownian motion, the distribution of X_1 is the Gaussian one. Lévy processes underlying distributions are in fact the class of infinitely divisible distributions – there is a correspondence between these distributions and Lévy processes. Therefore, if one wishes to build a Lévy process, one has to pick up a particular infinitely divisible distribution and construct the process accordingly.

It is important to notice that wide quantities of known distributions are infinitely divisible. Indeed, the lognormal, stable, *t*-student, gamma, Cauchy, Pearson VI distributions and many others are infinitely divisible. One can also cite the general extreme value distributions such as the Weibull or Gumbel ones... nearly all distributions owing support limited to a half line are infinitely divisible ; yet, distributions with bounded support (such as the uniform or binomial ones) will never share that property. From this paragraph we can conclude that it is possible to construct a big variety of Lévy processes ; the ones currently used in finance and the ones we introduce in the last section of this article being possible examples – amongst others.

In general, not all infinitely divisible distributions are explicitly known. The problem is then to find a proper characterization of the Lévy process. In fact infinitely divisible laws can be allocated a simple and tractable representation : the characteristic function (Fourier transform) of the distribution. The explicit expression of the characteristic function of an infinitely divisible distribution is called the *Lévy-Khintchine* formula and its determination dates back to the thirties. We recall this formula here :

$$\psi(u) = E(e^{iuX_t}) = e^{ibtu - \frac{ctu^2}{2} - t\int_{\mathbb{R}} [e^{iux} - 1 - iux1_{[-1,1]}(x)]\nu(dx)}$$

where :

$$\begin{cases} b \in \mathbb{R} \\ c \in \mathbb{R}^+ \\ \int_{\mathbb{R}} (x^2 \wedge 1)\nu(dx) < +\infty \text{ and } \nu(\{0\}) = 0 \end{cases}$$

We gave the formula for the infinitely divisible distribution of X_t. The formula, holding for any t, holds for the process and we will understand it as the underlying determinant of a Lévy process. Let us now analyze the various parameters appearing in the formula.

It can be easily seen that the parameter b corresponds to a linear drift (compute the Fourier transform of bt), but notice that this drift is dependent on the construction of the Lévy-Khintchine characteristic function. More precisely, one can replace the indicator function above by another similar function and this implies a change in b (not in c or ν). To consider b fully as a drift, an additional condition is required – we will detail this later on. The parameter c corresponds to a Brownian motion part in the Lévy process. The determination of the characteristic function of X_t in term of jumps is finally done by assigning a specification to ν, which is commonly called the Lévy density.

We recall two major criteria on Lévy processes : activity and variation. The activity is defined as the integrated value of the Lévy density :

$$\int_{-\infty}^{+\infty} \nu(dx)$$

Requiring finiteness of the activity is equivalent to requiring a finite gross arrival rate of jumps. The variation can be seen as a total absolute path followed by the process. For the process to be of finite variation there should be no Brownian part and the following should be respected :

$$\int_{|x| \leq 1} |x|\nu(dx) < +\infty$$

see Satō (1987) on page 140. This is indeed the constraint to be verified in order to define correctly the drift.

We conclude from above that there are basically two (related) ways to specify a Lévy process : either directly by a parameterization of the distribution or indirectly by a parameterization of the Lévy density. Therefore we will discriminate between distribution and Lévy-Khintchine based approaches. Of course, both approaches are often available for the same process. Our distinction does not rely on common mathematical classifications or on the structure of the processes, but on the way these processes may be used and specified by the modeler.

2. Modelling of stock prices

We may finally come to the description of the kind of modelling that is going to be used and developed in this paper. We recall beforehand the two standard ways of expressing price dynamics in the Black and Scholes model.

In a first approach, prices are modelled by making use of stochastic differential equations. In the particular lognormal framework this corresponds to assuming the following dynamics :

$$dS_t = S_t(\mu dt + \sigma dz)$$

In a second approach, prices are given explicitly such as :

$$S_t = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma z}$$

Therefore, when wanting to build a model using Lévy processes, one will start from one of these two approaches – which of course are related via Itō calculus. In the first approach, one would take stochastic differential equations like :

$$dS_t = S_{t^-} * (\mu dt + \sigma d\mathfrak{L}_t)$$

where $\mathfrak{L}$ is a Lévy process and S the modelled price dynamics.

This is the approach chosen in Chan (1999). The solving of this stochastic differential equation leads to Doléans-Dade or stochastic exponentials. In most publications, though, it is the second approach that is taken as a starting point. This article in particular reviews and aims at extending models assuming :

$$S_t = S_0 e^{\mathfrak{L}_t}$$

where $\mathfrak{L}$ is a general Lévy process possibly including a drift term and S the modeled price dynamics. This expression would justify the name "log-Lévy prices" since by taking the logarithm of the price S we recover a Lévy process $\mathfrak{L}$.

Finally, one needs to write :

$$S_t = S_0 e^{\mu t} \frac{e^{\mathfrak{L}_t}}{E(e^{\mathfrak{L}_t})}$$

when one wants to extract the drift (here μ) from the price dynamics.

III The distribution-based approaches

In the last section we briefly exposed what Lévy processes are and how they can be used to model the logarithms of price dynamics. At present we make a review of a first group of models that use Lévy processes in the representation of market prices. As explained before, when dealing with Lévy processes, one can start either from a particular distribution or from a particular characteristic function. The models studied in this section correspond to the first choice.

1. *The framework of generalized hyperbolic laws*

The first detailed Lévy models for finance were introduced in the mid nineties. They rely on generalized hyperbolic distributions and are an extension of Ole Barndorff Nielsen's work on wind blown sands in the late seventies. These models were built and developed because of their capability to furnish good fits to observed return distributions. In other words they can account for the asymmetry, leptocurticity... that govern real world distributions and make them differ noticeably from the simple Gaussian distribution. A detailed description of generalized hyperbolic laws and their applications to finance can be found in Prause (1999).

Generalized hyperbolic distribution's densities are of the following form :

$$\rho(x) = e^{\beta(x-\mu)} \frac{(\alpha^2 - \beta^2)^{\frac{\lambda}{2}} (\delta^2 + (x-\mu)^2)^{\frac{\lambda}{2} - \frac{1}{4}}}{\sqrt{2\pi}\,\alpha^{\lambda - \frac{1}{2}} \delta^{\lambda}} \frac{K_{\lambda - \frac{1}{2}}(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{K_{\lambda}(\delta\sqrt{\alpha^2 - \beta^2})} \quad (1)$$

where K denotes a modified Bessel function of the third kind.

These five parameter distributions provide a direct and good fit to price returns (see again the PhD thesis of Prause). They can be shown to be infinitely divisible : one can use them to build a Lévy process for the modelling of return dynamics. Direct intuition of the meaning of each parameter is not trivial, though there are some changes of parameterization that can yield proxies for skewness and kurtosis.

Notice that Lévy processes owning generalized hyperbolic distributions can in fact be built in another way by *subordination* that is by a Brownian time change. As far as laws are concerned, a generalized hyperbolic distribution is in fact a mixture of a Gaussian distribution by a generalized inverse Gaussian one. This latter class includes in particular the inverse Gaussian and gamma distribution (see the Variance Gamma model in the next section).

It is important to remark that the generalized hyperbolic characteristic function and moment generating function have similar simple forms (cf. Prause (1999)). These functions are particularly useful when it comes to the pricing of derivatives. A widely used approach is the Esscher transformation, which is a tool, coming from the insurance realm that allows building measure changes and relies strongly on the existence of the moment generating function. Also, one has to be aware of the following aspect : if X_1 is a generalized hyperbolic distribution, this is not a guarantee that a particular X_t will have the same type of distribution. Stability under convolution does not hold in the general case and this implies a need to use numerical methods if one wishes, as is often the case, to consider several different time horizons.

In this distribution approach, the derivation of an explicit formula for the Lévy density is *not* required. For the sake of completeness we give here the expression of the Lévy density with the restriction that λ be positive. Notice that this expression is particularly difficult to interpret intuitively (and this is worse in the general case) :

$$\nu(x) = \frac{\lambda e^{-\alpha|x| + \beta x}}{|x|} + \frac{e^{\beta x}}{|x|} \int_0^{+\infty} \frac{e^{-|x|\sqrt{2y + \alpha^2}} dy}{\pi^2 y[(J_\lambda \delta\sqrt{2y})^2 + Y_\lambda(\delta\sqrt{2y})^2]}$$

where J and Y are Bessel functions. We refer to Prause (1999) for a demonstration of this result.

To conclude on generalized hyperbolic laws, we recall the semi-heavy tail property followed by the asymptotic distributions :

$$\rho(x) \overset{\infty}{\simeq} |x|^{\lambda - 1} e^{-\alpha|x| + \beta x} \quad (2)$$

This seems to be a good compromise between *light* exponentially decreasing tails and say the *heavy* power law tails of α-stable processes.

2. Particularities of the hyperbolic model

The hyperbolic model has been derived by assigning simple hyperbolic laws to the returns. This is therefore a submodel of the one studied above and it has been studied in a financial context first by Eberlein *et alii* (1998). The hyperbolic distribution expression is obtained by choosing $\lambda = 1$ in (1). This distribution, having four parameters, avoids an over parameterization feature that can be obtained with generalized hyperbolic distributions. Notice that the logarithm of the hyperbolic distribution's density is a hyperbola ; this explains the name "hyperbolic".

Extensive studies on the German stock market proved that this model fits more to reality than the Black and Scholes model. Notice however that tails admit an exponential form. To see this one can just replace λ by one in (2). This light tail feature is often considered too sharp. Empirical tails are semi-heavy : lighter than the asymptotic Pareto tails of Mandelbrot models but heavier than the exponential tails of a Gaussian based model. Yet, the hyperbolic model may not be completely rejected ; in particular it takes into account the asymmetry and kurtosis aspects observed in empirical data.

3. Particularities of the Normal Inverse Gaussian model

The normal inverse Gaussian model is another submodel of the generalized hyperbolic model. This model is known for its increased tractability compared to the general case. It has been developed by Barndorff-Nielsen *et alii* (1997, 1998). Its name corresponds to the fact that the normal inverse Gaussian law is the mixture of a Gaussian law by an inverse Gaussian one.

The normal inverse Gaussian distributions correspond to setting $\lambda = -\frac{1}{2}$ in the formula (1) of the generalized hyperbolic distribution's density and consequently are four parameter based. The normal inverse Gaussian distributions were shown to provide a better fit to the data than the hyperbolic ones ; in particular they display non log-linear tails that admit a semi-heavy form (to see this replace λ by $-\frac{1}{2}$ in (2)).

These distributions, contrary to the hyperbolic ones, are stable by convolution. In other words, all the distributions attached to a normal inverse Gaussian Lévy process are of the normal inverse Gaussian type. This is most useful for the pricing of derivatives : in this context one may need a closed form expression for the distributions over various time scales. From what has been said, it is apparent that the normal inverse Gaussian model exhibits very nice features amidst the generalized hyperbolic family of models.

4. The Meixner Model

We end up this section with Meixner processes. This is another class of Lévy processes that has been introduced recently for the modelling of stock price dynamics by Schoutens (2002). This model corresponds to a distribution-based approach but it is not a subclass of the generalized hyperbolic model like in the two previous subsections. The driving Meixner distribution is :

$$\rho(x) = \frac{\left(2\cos\left(\frac{b}{2}\right)\right)^{2d}}{2a\pi\Gamma(2d)} e^{\frac{b(x-m)}{a}} \left|\Gamma\left(d + \frac{i(x-m)}{a}\right)\right|^2$$

with corresponding characteristic function :

$$E(e^{iuX_1}) = \left(\frac{\cos\left(\frac{b}{2}\right)}{\cosh\left(\frac{au-ib}{2}\right)}\right)^{2d} e^{imu}$$

Notice that the Lévy density for this model admits a quite simple form :

$$\nu(x) = \frac{d}{x}\frac{e^{\frac{bx}{a}}}{\sinh\left(\frac{\pi x}{a}\right)}$$

These processes are pure jump infinite variation ones ; their distributions possess as before semi-heavy tails. This model can be considered an alternative to the cases studied above : experimental tests done by Schoutens show that it succeeds quite well in the pricing of options (to be keen on the recovery of smile, it is possible to add up stochastic volatility to this Lévy model, as it is possible under most circumstances).

IV The Lévy-Khintchine based approaches

In this section we make a review of both the Variance Gamma and CGMY models, which can be seen to rely on a specification of the Lévy density. We study their Lévy-Khintchine based construction and comment on their characteristics.

1. The construction of Variance Gamma Processes

The introduction of Variance Gamma processes to model stock price returns dates back to Madan and Seneta (1990) and to Madan and Milne (1991).

These authors provided one of the first *integrated* frameworks where stock log prices are modelled *via* the use of a quite general class of Lévy processes. Following this work, extended Variance Gamma processes where built by Carr *et alii* (1998). They are now the core of what is known as the Variance Gamma model. Hence, when using the terminology "Variance Gamma", we will indeed refer to that extended framework. We detail both models in the following.

We examine here the construction of variance gamma processes. Take a drifted Brownian motion :

$$X(t) = \theta^* t + \sigma^* W_t$$

Then take a unit mean rate gamma process $\gamma(t, 1, v)$, that is, a Lévy process whose increments are distributed according to gamma laws and whose mean is set to unity.

To build Variance Gamma processes, one needs to *time change* the drifted Brownian motion by the unit mean rate gamma process, therefore yielding :

$$VG(t, \sigma, v, \theta) = \theta^* \gamma(t, 1, v) + \sigma^* W_{\gamma(t, 1, v)}$$

We note that the characteristic function and Lévy density corresponding to this process can be written as :

$$E(e^{iuVG_t}) = \frac{1}{\left(1 - i\theta v u + \frac{v\sigma^2 u^2}{2}\right)^{\frac{t}{v}}}$$

and

$$v(x) = \frac{e^{\frac{\theta x}{\sigma^2}} e^{\left(-\frac{|x|}{\sigma}\sqrt{\frac{2}{v} + \frac{\theta^2}{\sigma^2}}\right)}}{v|x|}$$

One can compute the four first central moments of the log price distribution from this expression of the characteristic function :

$$\begin{cases} m = t\theta \\ v = t(\theta^2 v + \sigma^2) \\ s = tv^{-\frac{3}{2}}(2\theta^3 v^2 + 3\sigma^2 \theta v) \\ k = 3 + tv^{-2}(3\sigma^4 v + 12\sigma^2\theta^2 v^2 + 6\theta^2 v^3) \end{cases}$$

The Variance Gamma processes that are constructed here thus depend on three parameters. The parameter σ, following standard conventions, corresponds to the volatility of the process. The parameter v allows for a quantization of the kurtosis. This can be intuitively understood : v corresponds to the variance rate in the time change process and it seems natural that a lot of agitation in the time change yields higher tails/kurtosis for the resulting process. Finally, the introduction of the third parameter θ corresponds to a need to account for the skewness of empirical distributions. Notice that in the first approach there was no θ parameter and hence skewness could not be taken into account ; in this two parameter model only volatility and kurtosis could be dealt with.

It is also important to notice that there exists a representation in law of Variance Gamma processes as the difference of two gamma processes. Indeed one can write in law :

$$VG(t, \sigma, v, \theta) \overset{law}{=} \gamma(t, \mu_p, v_p) - \gamma(t, \mu_n, v_n)$$

Both parts of this decomposition are subordinators, that is strictly increasing and positive Lévy processes. The first part corresponds to the upward movements of the dynamics whilst the second part corresponds to the downward movements. Hence this is a particularly simple and intuitive representation of Variance Gamma processes. It is remarkable that the process is described in terms of four parameters – two mean rates and two variance rates – whereas the original process is built with three parameters : there exists necessarily a constraint between these four parameters.

This simple representation in terms of gamma processes brings along a simple representation for the Lévy density of the Variance Gamma process :

$$v(x) = \begin{cases} \left[\frac{\left(\mu_n^2 e^{-\frac{\mu_n}{v_n}|x|}\right)}{(v_n|x|)}\right] & \forall x < 0 \\ \frac{\mu_p^2 e^{-\frac{\mu_p}{v_p}|x|}}{v_p|x|} & \forall x > 0 \end{cases}$$

Notice that this expression of the Lévy density as a ratio of a decreasing exponential by a power function will be used in the following. The consequence of taking such forms of Lévy densities is that constructed processes are of infinite activity and finite variation.

We conclude on the representation given above : one of the constructions of variance gamma processes starts from a particular form of the Lévy density. The denomination "Lévy-Khintchine" or equivalently "Lévy density" based approach finds its justification in that case, when one wants to insist on the role played by the Lévy density as a starting point. We insist that our classification is not mathematical in essence, for if it were, we would have included the variance gamma model into the generalized hyperbolic one (variance gamma distributions being mixtures of Gaussian by gamma distributions which are themselves particular cases of generalized inverse Gaussian distributions). Our classification opposes modelling approaches based on distributions to approaches based on Lévy densities. Notice in particular that there might be classes of Lévy processes with no explicit or reasonably tractable underlying standard distributions : it is yet always possible to start from the Lévy density. We detail this in the following and introduce a model directly and uniquely based on a Lévy density specification.

2. *An extension via the Lévy density : the CGMY model*

In the footsteps of the Variance Gamma model an extension emerged soon. This extension is a new and more general model which has been called the CGMY model according to the names of its four authors : Carr *et alii* (2002). The CGMY model differs from the other models developed beforehand : it also provides for Lévy processes to serve as log prices dynamics, but it is not based on a specification of the log price densities or on a time changed Brownian motion. The originality of this model is in its direct specification of the characteristic function of the Lévy process – in other words in the direct choice of a particular form of Lévy density. The choice of Lévy density characterizes the model and is the root for all subsequent calculations.

In the CGMY model, the Lévy density is hence supposed to take the following form :

$$v(x) = \begin{cases} \frac{Ce^{-G|x|}}{|x|^{1+Y}} & \forall x < 0 \\ \frac{Ce^{-M|x|}}{|x|^{1+Y}} & \forall x > 0 \end{cases}$$

With such a choice, an explicit calculation of the characteristic function can easily be led :

$$E(e^{iuCG_t}) \;=\; e^{tC\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y]}$$

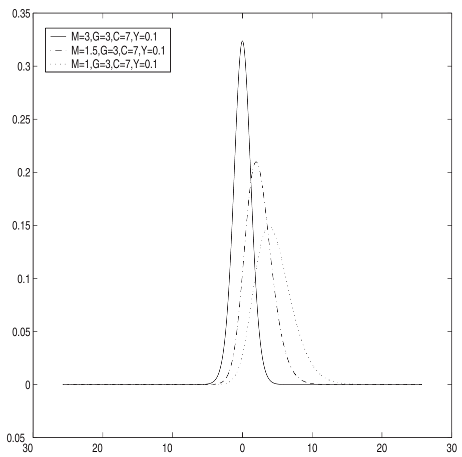
CG_t standing for the CGMY Lévy process which is of infinite activity and finite variation when Y takes values in $]0,1[$.

Notice that the associated density of the Lévy process can be obtained by numerical inversion of the characteristic function (in other words by a fast Fourier transform). The four first moments, in the case of a *pure* (without any added diffusion part) finite variation CGMY process can be expressed as :

$$\begin{cases} m \;=\; tC\Gamma(1-Y)\left[\dfrac{1}{M^{1-Y}} - \dfrac{1}{G^{1-Y}}\right] \\ v \;=\; tC\Gamma(2-Y)\left[\dfrac{1}{M^{2-Y}} + \dfrac{1}{G^{2-Y}}\right] \\ s \;=\; tv^{-\frac{3}{2}}C\Gamma(3-Y)\left[\dfrac{1}{M^{3-Y}} - \dfrac{1}{G^{3-Y}}\right] \\ k \;=\; 3 + tv^{-2}C\Gamma(4-Y)\left[\dfrac{1}{M^{4-Y}} + \dfrac{1}{G^{4-Y}}\right] \end{cases}$$

It appears that the C and Y parameters provide a gross control on all moments. The parameters G and M quantify the asymmetry of the Lévy density. When G is bigger than M, the Lévy density is skewed to the right (its left part is crushed due to the exponential factor). To this asymmetry of the Lévy density corresponds an asymmetry of the return distribution ; it is for example also skewed to the right $(s \geqslant 0)$ when G is bigger than M. This point is illustrated in figure 4 where distributions have been computed by FFT inversions of the characteristic function :

Figure 4. **CGMY distributions : impact of M/G on the skewness**



If one takes a look at the Lévy density, one will conclude that the parameter Y does not have an impact on the skewness of the return density. Yet, this is wrong : for an already skewed density, a change in Y will yield a change in skewness. This can be deduced from the expression of s. In *figure 5*, we plot return distributions (computed by FFT inversion of the characteristic function) for various values of Y.

Using a database of daily Alcatel stock prices on a five-year period, we conducted an empirical study to monitor how the CGMY model applies to reality. We determined the parameters C, G, M and Y for Alcatel by two different methods : maximization of the likelihood and minimization of the quadratic discrepancy between the empirical and model distributions. Both methods need inversions of the characteristic function by Fourier transform. We found out that the parameters estimated by the two methods are significantly the same. The results are summed up in figure 6 and 7 where the empirical returns, the CGMY fit of these returns and also a normal fit are plotted. The log-likelihood obtained in the first case is of 982 whereas it is of 958 in the second case. These figures are here to illustrate that the CGMY approach allows for a much better description of reality than the standard Black and Scholes framework : the central peak and thickness of tails are well rendered, just as the skewness (or say $M > G$) effect that translates here into a thinner right tail. For a statistical analysis, see Carr *et alii* (2002).

Figure 5. **CGMY distributions : impact of Y on the skewness**

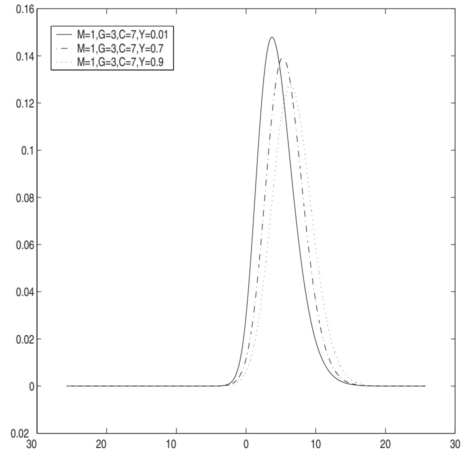


Figure 6. **Fitting of Alcatel returns by a Gaussian distribution**

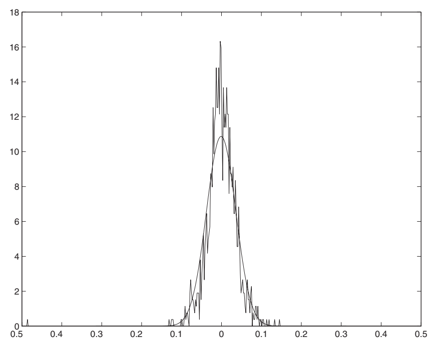
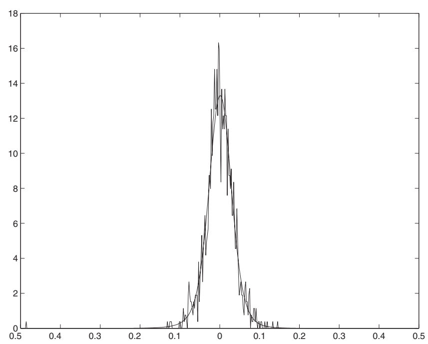


Figure 7. **Fitting of Alcatel returns by a CGMY distribution**



As for the tails, they can be expected to be neither too heavy nor too light, due to the expression given *ad hoc* to the Lévy density in the Variance Gamma, CGMY and subsequent models. For example, it is well known that for stable Lévy processes there is equivalence between the Lévy density tail and the distribution tails (with Pareto power law decreasing form).

V Extending the CGMY framework

We introduce in this section new Lévy dynamics that can be considered part of a family containing Variance-Gamma and CGMY processes and that allow for an important quantity of average size jumps.

1. *Expected features*

It appears from the preceding sections that various Lévy processes have been suggested to help modelling financial stock prices. Some models concentrate on the distributions when others start from the *Lévy-Khintchine* formula. We want to study more thoroughly this second class of models. In particular the question of the good choice of the Lévy density appears to be crucial.

From their empirical investigations, Carr *et alii* deduce that the Lévy processes underlying stock prices should be of finite variation and infinite activity. Confirmation of these hypotheses seems quite sound. Indeed, one expects market prices to display an important arrival rate of small movements, which is equivalent to requiring infinite activity. Also, since the numerous small movements are modelled directly through the Lévy jump process, there should be no need for a Brownian diffusion, which is the ideal case owning continuous paths – perfect continuity not existing in the real world. Finally, finite variation accounts for a certain amount of regularity in the process. The processes chosen in the following will thus be of infinite activity and finite variation.

In the CGMY model, Lévy densities are completely monotonic. This means that there is a decreasing relationship between the size of the jumps and their importance measured via the Lévy density. This hypothesis can of course be relaxed in order to have more flexibility in the choice of shape of Lévy density. Nevertheless, we believe that there are good reasons – apart from flexibility –to relax this hypothesis.

The Lévy density models (instantaneous) jumps ; it should be therefore strongly dependent on the behavior and psychology of market participants. We believe that individuals “visualize” prices through levels. For example : one euro, ten euros, fifteen euros... are price levels that correspond to common representations. Similarly, if new information arrives on a particular company, traders should mentally increase or decrease the price of the company's stock by a certain level, this act being conscious or not. It seems reasonable to assume that these levels are not numerous. For instance and with a bit of simplification : to good news / anticipations on the sector of the stock's company could correspond a one euro increase, to good news on the company a two euros increase, to very good news a five euros increase and so on.

These mental representations should translate in practice in quite determined up/down bumps on the traded security price, in other words jumps should occur around privileged levels. This is in contrast to the complete monotonicity hypothesis and this is the reason why we think this hypothesis should be relaxed. As far as returns are concerned, they correspond to an aggregation of jumps over a defined, possibly small, period. Their distribution, due to an averaging/mixing effect, need not be multimodal contrary to the Lévy density.

One can anticipate return distributions to mimic the Gaussian distribution for large density time steps (aggregational Gaussianity), whereas they could possibly display multimodality over very small time steps. To these expected features are associated particular Lévy processes in the following two subsections (in the first one, the furnished Lévy density is over-simplified for demonstrative purposes).

2. *An illustration with compound Poisson processes*

In this subsection, we aim at exhibiting simple Lévy processes whose Lévy densities are not completely monotonic. We thus build a compound Poisson process owing the simple Lévy density given in *figure 8*. This density has a main peak about zero and second order lateral peaks. The corresponding process should display many small moves and a reasonably high quantity of average size moves – which is not the case if one takes the complete monotonicity as an assumption.

Figure 8. **Lévy density**

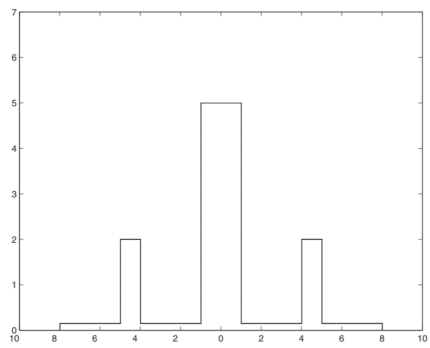
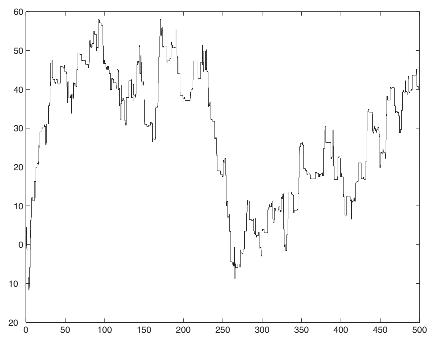
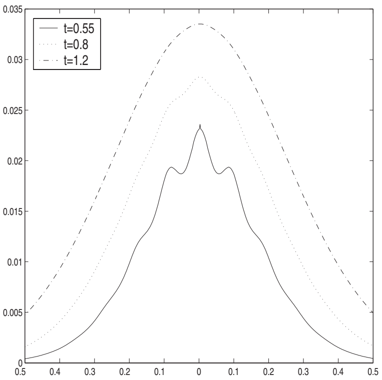


Figure 9. **Paths of the compound Poisson process**



The paths of such a process can be drawn in a standard way : firstly, simulate a Poisson process with intensity the integrated value of the Lévy density ; secondly simulate jumps with distribution the Lévy density renormalized by its integrated value. The result has been plotted in *figure 9*.

Figure 10. **Distributions at various time scales**



If one takes a close looks at the paths of this process, one can indeed notice the quite high quantity of jumps with size comprised between four and five as could be deduced from the shape of the Lévy density. This situation is of course exaggerated and was chosen for its illustrative purpose.

The characteristic function of this Lévy process is of the following form[1] :

$$E(e^{iuX_t}) = e^{t(9.7\,\mathrm{sinc}(u) - 14.8\,\mathrm{sinc}(4u) + 19.5\,\mathrm{sinc}(5u) + 2.4\,\mathrm{sinc}(8u) - 15.8)}$$

From this expression, one can deduce by a Fourier transformation the shape of the distribution function of the returns at different time scales (*figure 10*). In accordance to what was anticipated in the previous subsection, it is possible to have at the same time a monotonous distribution and a non-monotonous Lévy density. Notice also that an increasing t yields a shape closer and closer to the Gaussian shape. On the contrary, for a small enough t, the distribution becomes multimodal and, indeed, its shape resembles more and more the Lévy density shape.

In the following subsection, we show how to build a more realistic process with non-monotonous Lévy density for the modelling of stock log-prices.

3. *α-β Lévy motions*

In the previous study a simple compound process was built to demonstrate how to account for a possibly high quantity of medium size jumps. Here, we want to construct a more realistic process for the modelling of the returns. We remain in the Lévy-Khintchine based framework and start from the Lévy density of the CGMY model. This density is modulated with an additional simple factor in the following way :

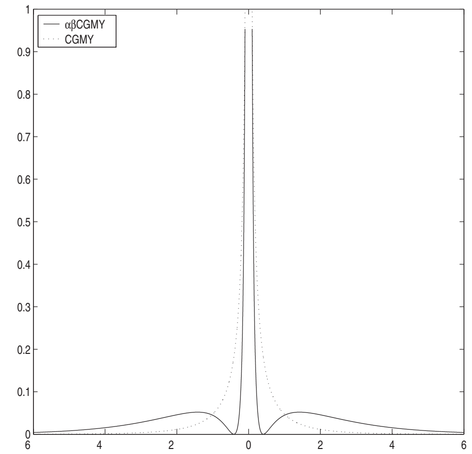
$$v(x) = \begin{cases} C\dfrac{e^{-G|x|}}{|x|^{1+Y}}(\alpha - |x|^{\beta})^2 & \forall x < 0 \\ C\dfrac{e^{-M|x|}}{|x|^{1+Y}}(\alpha - |x|^{\beta})^2 & \forall x > 0 \end{cases}$$

Notice that is it a possibility amongst many others. This new density is dependent on the traditional C, G, M and Y factors plus the modulating α and β factors. Its shape has been graphed in *figure 11* (along with the CGMY reference in dotted line) :

With this Lévy density, the main difference with the CGMY model is the presence of two humps modelling a high quantity of medium size jumps (the left hump corresponding to downward moves and the right hump to upward moves).

We give this example because of its simplicity. It is also possible to extend the CGMY density in a way to get more humps – to model several jump levels of preference for the investors – or in a way never to touch the abscissa (for if the Lévy density does it, then there is a level of jump that is never attained) by introducing sine factors in the Lévy density for instance.

Figure 11. ***α-β* extended CGMY Lévy density**



In our case the characteristic function writes[2] :

$$E(e^{iuX_t}) = \begin{array}{l} e^{t\alpha^2 C\Gamma(-Y)([M-iu]^{Y} - M^{Y} + (G+iu)^{Y} - G^{Y}]} \\ * e^{tC\Gamma(2\beta - Y)[(M-iu)^{Y-2\beta} - M^{Y-2\beta} + (G+iu)^{Y-2\beta} - G^{Y-2\beta}]} \\ * e^{-2t\alpha C\Gamma(\beta - Y)[(M-iu)^{Y-\beta} - M^{Y-\beta} + (G+iu)^{Y-\beta} - G^{Y-\beta}]} \end{array}$$

This characteristic function can be inverted numerically to graph the distributions at various time scales, see *figure 12*. The shorter the time scale, the more humped the distribution. Note that these curves have been graphed upon renormalization and with β corresponding to a finite activity assumption.

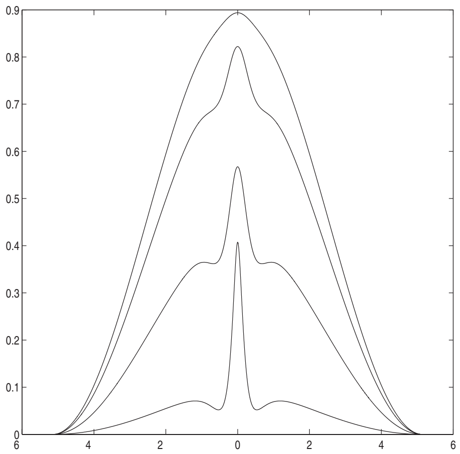
It is also possible to get analytical expressions for the moments. We only provide for the second moment (the other moments can be computed in a similar way) :

$$\begin{aligned} v = {} & t\alpha^2 C\Gamma(2-Y)\left[\frac{1}{M^{2-Y}} + \frac{1}{G^{2-Y}}\right] \\ & + tC\Gamma(2+2\beta-Y)\left[\frac{1}{M^{2+2\beta-Y}} + \frac{1}{G^{2+2\beta-Y}}\right] \\ & - 2t\alpha C\Gamma(2+\beta-Y)\left[\frac{1}{M^{2+\beta-Y}} + \frac{1}{G^{2+\beta-Y}}\right] \end{aligned}$$

To conclude, it is possible to get explicit mathematical expressions for all the standard useful tools except for the return distributions. The pricing of options relies strongly on the Fourier transform method, whether one adopts the Esscher or Madan methodologies related to the setting of a particular risk-neutral world (see Le Courtois and Quittard-Pinon (2003) for more details).

The model proposed here allows taking into account humped distributions at short time scales, where amonotonicity of the Lévy density theoretically induces amonotonicity of the return distributions. This theoretical feature is interesting in that it allows for a modelling of an empirical feature that is observed on the market, over various time horizons. In essence, this model is a link between all time scales ; and one can envisage calibrating the model indistinctly by starting from intraday (whatever the frequency) or daily data. What is important to notice is that distributions are allowed to take simultaneously various forms according to the time scale. This is in deep contrast to the stable approach of Mandelbrot in which the same distributions appear at all horizons. As a conclusion, the underlying mechanisms on the market should differ with the time scale, just as the empirical data do, and the mathematical tools should also be built accordingly.

Figure 12. **Remormalized distributions of the extended model**



VI Conclusion

In this article we first recalled main facts about Lévy processes : their defining property of independent and stationary increments, the fact that most often they are pure jump processes when no Brownian part is envisaged, their description through the Lévy-Khintchine formula... We also recalled how they can be used in exponentials for the modelling of stock prices (or directly for the modelling of returns).

Existing models have been classified according to the object they aim at describing and using in priority : return distributions or characteristic functions. The first class of models contains the generalized hyperbolic models, its sub parameterizations that are the NIG and hyperbolic models, and the Meixner model. In the second class can be included the Variance Gamma model (when taken from the point of view of the Lévy density) and especially the CGMY model.

In the final section, we give examples of models that relieve the monotonicity hypothesis on the Lévy density. This furnishes processes that can be used in the study of markets where participants have localized preferences in terms of price jumps. A simple compound Poisson model is first given, followed by a more complex model that helps extend the CGMY framework.

This study is limited to the application of Lévy processes to the modelling of stock returns but can as well be extended to the study of other financial products like bonds for example. Recall that the aim was to obtain processes that are more suited to finance than the Brownian motion but no so specific as to be useful only in particular domains of finance. One important issue of finance is the pricing of derivatives products. In a Lévy framework, the risk-neutral measure is not unique and its determination is the main problem at stake. Two main solutions have been proposed : use of an Esscher transform, direct determination of the measure by calibration on option prices.

A model based on Lévy processes, contrary to the Black and Scholes model, permits to recover partly the smile. Lévy processes render the jumps observed in financial dynamics. Another important empirical feature is the existence of boosted and flat periods in the dynamics. Two main approaches allow dealing with this feature : the GARCH (see Hull (2000) for an application of GARCH (1,1) to finance) and the stochastic volatility approach (see for example the model of Heston (2003)). A stochastic volatility model, just like a Lévy based model, allows rendering part of the smile effect on derivative products. It has appeared that with a combined framework, the smile is completely recovered ; see Carr *et alii* (2001) for the introduction of such a new framework.

Notice in passing that the advantage of this large framework proposed by Carr *et alii* is that the smile can then be *analyzed* as the price of two different things : a premium for the possibility of instantaneous large jumps associated to a premium for the possibility of volatility boosts. We conclude by saying that the increasing amount of articles devoted to the application of general Lévy processes to finance seems to be an indication that this approach will be central to the financial and possibly actuarial research of the coming decade. ■

1 Hint to compute this characteristic function : use the fact that $\int_{-\lambda}^{\lambda} (e^{iux} - 1)dx = 2\lambda(\mathrm{sinc}(\lambda u) - 1)$ where *sinc* stands for cardinal sine.

2 Decomposing the Lévy density, as can do this :
$\nu = \mu(\alpha^2 C, G, M, Y) + \mu(C, G, M, Y - 2\beta) + \mu(-2\alpha C, G, M, Y - \beta)$
where μ is the CGMY Lévy density.

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