Aggregation and Credit Risk Measurement in Retail Banking

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Basel II agreement on banking supervision is to be completed in the coming months with implementation due 2006. Besides the interest devoted to operational risk, the Basel II Agreement relies in a multiple step approach for credit risk measurement with different degrees of sophistication. The most "advanced" is known as *Internal Ratings Based* approach (IRB) which is to be implemented by large international banks for credit risk capital charges computation. It appears as a technological revolution as this is the first time quantitative methods will be used for credit in a regulatory framework.

Credit risk modelling [1] in the IRB setting is similar to CreditMetrics or Moody's-KMV. The focus is drawn on the modelling of the loss distribution in the credit portfolio, achieved through the use of default indicators at a given time horizon. This approach relies on the econometrics of categorical variables, being used for thirty years in credit scoring [2]. In comparison to market risk, the correlation modelling [3] is quite simple, since it is assumed that there is a unique risk factor driving all the credit market. This hypothesis is questionable, in particular for international banks [4]. It is not acceptable for economic reasons, but can be seen as a way to ease subsequent capital computations. Indeed, capital requirements for well-diversified credit portfolios become linear function of the amount lent to borrowers. In other words, risk measures become additive. We intend in this article to evaluate the reduction of capital requirements due to credit risk diversification. To proceed, we propose a more general model than Basel II, still more simple than CreditMetrics or KMV, where the correlation between factors is not necessarily equal to 100%. We then express losses on retail portfolios, relying upon a typical "small-risk" hypothesis, though our analysis can be applied to corporate borrowers as well.

The second ingredient in the IRB approach is the quantification or measure of risks with quantiles of the loss distribution at a given time-horizon. It is mainly an extension of the Value at Risk methodology from market risk to credit risk. In order to assess the impact of the Committee's choice, we compare capital requirements computed with VaR or with the "Expected Shortfall" [5]. Another specificity of the measurement of credit risk within Basel II is the distinction between "expected loss" and "unexpected loss". For corporate credits and mortgages, the regulatory risk measure is based on the total loss. For retail credits (apart mortgages), the risk measure is based on the credit losses minus their expectation (or "expected loss"), this quantity being the "unexpected loss". As it will be seen further, this is a key issue, especially as far as risk contributions and capital allocation are concerned.

The article is organized as follows: in the first part, we review credit risk modelling underlying the Basel II agreement. We highlight our model as an extension of the regulatory one. The second part is devoted to risk measurement and capital requirements. We quantify credit risk reduction implied by diversification and the impact of the risk measure choice. We draw a case study based on a typical retail credit portfolio. We examine on one hand the

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effects on the overall risk of the bank, and on the other hand the risk contributions of the different subportfolios.

I Credit risk modelling

Credit risk modelling in the Basel II framework

In the framework proposed by the Basel Committe, each credit is identified by three parameters, namely the *Exposure at Default* (EAD) which is the credit outstanding capital one year ahead, the *Loss Given Default* (LGD) which represents the unrecovered amount in case of default, and the *Probability of Default* (PD). Banks pool their credit in homogeneous portfolios (mortgage credits, small borrowers credits, loans to small and medium enterprises) corresponding to their different operating activities. Depending on the sector and on the quality of data reporting, borrowers may also be pooled according to the origination year (vintage) and an internal rating. In the most advanced IRB setting, those parameters are determined by the banks following a methodology validated by the supervisory authorities [6]. When the credit characteristics (EAD, LGD, PD) and the sector of activity are identical, we deal with an "homogeneous portfolio".

The occurrence of default is represented by a Bernoulli random variable which takes the value 1 in case of default and 0 otherwise. Let us denote $Y_{J,i}$ this variable where *i* refers to the credit and *J* to the homogeneous portfolio it belongs to. The Basel II model, as CreditMetrics, makes use of a latent variable which follows a standard Gaussian distribution to represent $Y_{J,i}$. More precisely,

$$Y_{J,i} = \begin{cases} 1, \text{ if } (Z_{J,i} < s_J) \\ 0, \text{ if } Z_{J,i} \geq s_J \end{cases}$$

where $Z_{J,i}$ follows a standard Gaussian distribution which can be written:

$$Z_{J,i} = \sqrt{\rho_J}\Psi_J + \sqrt{1-\rho_J}\varepsilon_{J,i},$$

$\varepsilon_{J,i}$ and Ψ_J are independent random variables following standard Gaussian distribution. $\varepsilon_{J,i}$ represents specific risk to credit i and Ψ_J a common risk to all credits in the portfolio *J*. ρ_J is a correlation parameter ranging from 0 to 1, which allows one to take into account various degrees of dependence between default events depending on the portfolio. This correlation parameter is not estimated in principle by the banks but computed according to a regulatory formula (it depends mainly on the probability of default and on the kind of credit).

This factor model has been pioneered by Vasicek (1997) and studied by Gordy (2000, 2002). Let us point out that the default indicators $Y_{J,i}$ are not independent, due to the common factor Ψ_J, but conditionally on Ψ_J, default indicators $Y_{J,i}$ are independent. The underlying structure being Gaussian, the copula function (or dependence function) associated with the default indicators $Y_{J,i}$ is Gaussian as well (see below).

The threshold s_J is determined from the probability of default, PD_J of the credits within homogeneous portfolio *J*:

$$PD_J = P(Y_{J,i} = 1) = P(Z_{J,i} < s_J) = \Phi(s_J),$$

where Φ is the cumulative distribution function of the standard Gaussian distribution. Hence, $s_J = \Phi^{-1}(PD_J)$.

In the homogeneous portfolio, the joint probability of borrowers i and j defaulting at the time horizon is:

$$P(Y_{J,i} = 1, Y_{J,j} = 1) = P(\sqrt{\rho_J}\Psi + \sqrt{1-\rho_J}\varepsilon_{J,i} < \Phi^{-1}(PD_J), \sqrt{\rho_J}\Psi + \sqrt{1-\rho_J}\varepsilon_{J,j} < \Phi^{-1}(PD_J))$$

That is:

$$P(Y_{J,i} = 1, Y_{J,j} = 1) = \Phi_2(\Phi^{-1}(PD_J), \Phi^{-1}(PD_J) ; \rho_J)$$

We may question the presence of a single risk factor Ψ_J to model the dependence structure between credits in the portfolio *J*. Indeed, this single factor structure comes almost automatically from the homogeneity of the credit portfolio (see box 1 on de Finetti's theorem). Thus the one factor approach seems unavoidable for homogenous portfolios [7]. Even if we had used a multiple factor model at this stage, sensitivities would have been the same and we could have reformulated the model in a one factor setting.

1. Factor models for homogeneous portfolios

Def.: (*Bernoulli mixture*). Let p < m and a p-dimensional random vector $\Lambda = (\Lambda_1, \ldots, \Lambda_p)$. The random vector $Y = (Y_1, \ldots, Y_m)$ follows a Bernoulli mixture model with factor vector Λ if there are functions $Q_i : R^p \to [0, 1]$, with $1 \leq i \leq m$, such that conditionally on Λ the default indicators Y is a vector of independent Bernoulli random variables with $P(Y_i = 1|\Lambda) = Q_i(\Lambda)$.

Def.: (*exchangeability*). A random vector is exchangeable if any rearrangement of its components yields the same joint distribution.

Let us assume that the Bernoulli random variables which represent the default of each retail obligor are exchangeable. Then, we have de Finetti's theorem (see Frey et al.):

Theorem: Let $Y = (Y_i)_{i \geq 1}$ be an infinite sequence of Bernoulli random variables. Then there exists a probability distribution *G* on [0,1] of a one dimensional factor such that for every $k \leq m$, there is:

$$P(Y_1 = 1, \ldots, Y_k = 1, Y_{k+1} = 0, \ldots, Y_m = 0) = \int_0^1 q^k(1-q)^{m-k} dG(q)$$

Basel II model happens to be a particular case of de Finetti's results, assuming we have an infinity of borrowers. In particular, Basel postulates that the common factor is unidimensionnal and follows a standard Gaussian distribution. Relaxing the Gaussian assumption of the factor might seem necessary for taking into account extreme events. Still, empirical findings showed that retail banking could postpone further investigation in that direction for the moment. Our current issue is to address the quality of data and the reliability of parameter estimation.

Loss distribution in a large portfolio

The loss in the portfolio *J* is obtained as the sum of losses on individual credits:

$$L_J = \sum_{i \in J} EAD_{J,i} \times LGD_J \times Y_{J,i},$$

where $EAD_{J,i}$ and LGD_J stand respectively for the "exposure at default" and "loss given default" of the credits within the homogeneous portfolio J. The loss distribution for a homogeneous portfolio can be obtained by inversion of Fourier transform, since the characteristics function takes an explicit expression (see appendix). In the case where the amount of each credit is small, it is possible to approximate the loss in the portfolio in a tractable way. More precisely, we suppose that the overall exposure to credit risk in portfolio J remains constant $\sum_{i \in J} EAD_{J,i} = EAD_J$ and we let the amount of each credit tend to zero. It is the asymptotic approach which underlies the "Arbitrage Pricing Theory" (see Chamberlain et Rothschild (1983)). We can prove that almost surely (see appendix):

$$\sum_{i \in J} EAD_{J,i} \times LGD_J \times Y_{J,i} \rightarrow L_J(\Psi_J)$$

$$= EAD_J \times LGD_J \times \Phi\left(\frac{\Phi^{-1}(PD_J) - \sqrt{\rho_J}\Psi_J}{\sqrt{1-\rho_J}}\right)$$

The limit case corresponds to the so-called *infinitely granular* portfolio, according to Basel II terminology. We notice that each borrower's specific risk has been diversified away and that the remaining risk in the portfolio depends only on the factor Ψ_J. This approximation is valid for retail credit lines with more than 1000 borrowers [8]. Let us remark that due to the dependence between defaults events, there remains some risk even for infinitely granular portfolio.

Aggregated losses

Let us assume the overall credit portfolio of the bank has been split into K infinitely granular homogeneous portfolios. The loss for the overall bank, or aggregated loss L, can be written as the sum of the losses on each portfolio:

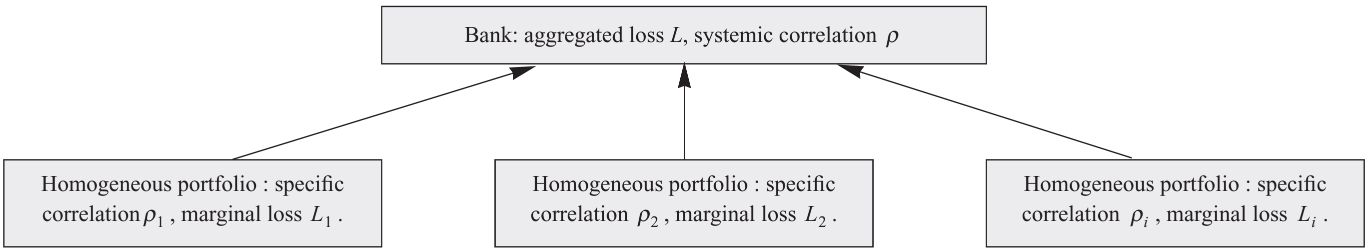
$$L = \sum_{J=1}^{K} EAD_J \times LGD_J \times \Phi\left(\frac{\Phi^{-1}(PD_J) - \sqrt{\rho_J}\Psi_J}{\sqrt{1-\rho_J}}\right)$$

Let us remark that at this stage we have not made any hypothesis on the correlation structure between the factors Ψ_J associated to the different credit portfolios. The Basel Committee takes for granted that the correlation is equal to 100% between the different risk factors. In other words, all the factors Ψ_J are equal. Thus, if there is some credit risk diversification in each portfolio, it is implicitly *postulated* that there is none between the different portfolios.

This is an hypothesis that we propose to retrieve in order to assess its practical importance. In the sequel, we suppose that there exist a common economy wide risk factor Θ, which can be understood as a systemic risk, and portfolio specific risk factors Θ_J such that we can write:

$$\Psi_J = \sqrt{\rho}\Theta + \sqrt{1-\rho}\Theta_J$$

where ρ stands for the systemic correlation and Θ as well as Θ_J are random variables following independent standard Gaussian distribution. ρ ranges from 0 up to 100%. In the Basel II framework, this correlation is equal to 100%. In the sequel, we will refer to the Committee's model as the "single factor" model. Here ρ can be estimated with historical loss data on each portfolio, recorded internally by the bank [9]. Our model has the advantage of simplicity because it needs only one extra parameter from the regulatory one. Let us notice that Moody's-KMV uses a more general correlation structure between the factors Ψ_J but it is difficult to estimate in the case of retail banking. We will not discuss the link between Θ, Θ_J, observable variables and other risk factors used by KMV-Moody's to assess credit risk for large corporate borrowers. This should be studied in order to aggregate the retail and corporate portfolios.



- The aggregation principle-

The aggregation principle displayed above shows the diversification effect we are willing to take into account. In the Basel II model, loss distribution in the homogeneous portfolios are comonotonic, i.e. monotone (decreasing) functions of the same factor. It means intuitively that a worst case for one portfolio is a worst case for all the other portfolios, which may be doubted in practice. Hence our willingness to propose the introduction of the correlation ρ at the aggregated level.

II Risk measurement and capital requirements

Context

As it is already the case for market risks in the IRB approach, bank capital charges must match the credit risk magnitude through use of an appropriate credit risk measure.

Recall that a risk measure is an indicator computed from the loss distribution. The risk measures retained by the supervisory authorities are VaR based. Those risk measures are often criticized, namely because in the case of default risk they are not sub additive. They do not take into account the magnitude of large losses as well. In order to better understand the different issues, we analyse credit risk of a retail bank with our model and with the IRB model as well. We have chosen to perform a case study, our example being typical of this sector.

Risk measurement in the Basel II framework

Let us start our analysis with a few reminders on risk computations in the Basel II setting and on the underlying model. If X is a random variable and $\alpha \in [0, 1]$, we define the lower quantile of order α as being $q_\alpha(X) = \inf(t, P(X \le t) \ge \alpha)$. In the Basel II case, X may either represent credit losses L, as for corporate credits and mortgages or unexpected losses, namely $L - E^P[L]$ for retail credits apart mortgages. For retail credits, expected losses $E^P[L]$ are supposed to be covered by the credit margin and banks do not have capital charges with respect to those losses. A simple reasoning gives that $q_\alpha(L - E^P[L]) = q_\alpha(L) - E^P[L]$ [11]. The expected loss on the aggregated portfolio is the sum of the expected losses in the sub-portfolios, namely $E^P[L] = \sum_{j=1}^{K} EAD_J \times LGD_J \times PD_J$. Concerning corporate borrowers and mortgage credits, the regulatory risk measure is $q_\alpha(L)$ and not $q_\alpha(L) - E^P[L]$. Thus, the Basel Committee considers that the margins on the credits granted to corporate borrowers and on mortgage credits do not cover the average risk, e.g. the expected loss. We will thereafter present risk measures based on unexpected losses, since the transposition to the total losses is straightforward.

The capital charges required to cover retail credit risk [12] in the Basel II setting must be at least equal to:

$$\zeta = q_\alpha\left(\sum_{J=1}^{K} EAD_J \times LGD_J \times \Phi\left(\frac{\Phi^{-1}(PD_J) - \sqrt{\rho_J}\Psi_J}{\sqrt{1-\rho_J}}\right)\right) - \sum_{J=1}^{K} EAD_J \times LGD_J \times PD_J$$

In the sequel, we will systematically make the assimilation between capital charges and risk measures.

In the IRB approach (single factor model) $\Psi_1 = \ldots = \Psi_K = \Theta$, corresponding indeed to the case where the systemic correlation ρ is equal to 100%, the risk measure takes a very simple additive form, thanks to the comonotonic additivity of quantiles:

$$\zeta = \sum_{J=1}^{K} EAD_J \times LGD_J \times \left[q_\alpha\left(\Phi\left(\frac{\Phi^{-1}(PD_J) - \sqrt{\rho_J}\Theta}{\sqrt{1-\rho_J}}\right)\right) - PD_J\right]$$

Here the risk measure in the aggregated portfolio is equal to the sum of the risk measures of the sub-portfolios [12]. This justifies the *weighting* approach in Basel II. Unfortunately, such a decomposition of the aggregated risk measure is not possible in our extended model, where the correlation between factors is not perfect anymore ($\rho < 1$). In the multifactor framework, it is still possible to estimate the risk measure, by Monte Carlo simulation or quantile computation from the empirical distribution [13].

Whereas the regulatory risk measure is no more than a recentered quantile of the loss distribution, it is possible to contemplate other risk measures. If X is a random variable with a positive density (which is verified in our study where we only consider infinitely granular portfolios), we define

$$ES_\alpha(X) = \frac{1}{1-\alpha}\int_{q_\alpha(X)}^{+\infty} x\,dP^X(x) = E^P[X | X \ge q_\alpha(X)],$$

where P^X is the distribution of X ($ES_\alpha(X)$ is the "Expected Shortfall" of X), then the risk measure $\kappa = ES_\alpha(L - E^P[L])$ [14]. κ is equal to the expectation of the losses in excess of the regulatory risk measure $q_\alpha(L - E^P[L])$, which allows us to establish immediately that $\zeta \le \kappa$, e.g. the second risk measure is more conservative than the regulatory one. We remark that as well as for the regulatory risk measure that $\kappa = ES_\alpha(L) - E^P[L]$. It is a coherent risk measure [15]:

$\kappa(L) \le \sum_{J=1}^{K} \kappa(L_J)$, where $L = \sum_{J=1}^{K} L_J$ and the L_J are the losses on the different credit portfolios. The risk measure that we have introduced, built from the "Expected Shortfall" of the loss distribution is comonotonic additive [16] and invariant in distribution. It belongs to the class of "spectral risk measures" or the "convex" distortion risk measures (cf. Dhaene et al. (2003)). These risk measures are being investigated by large banks as a reliable alternative to the regulatory measure. Since the risk measure based on Expected Shortfall being comonotonically additive, we obtain a very simple decomposition of the risk aggregated in the single factor model:

$$\kappa = \sum_{J=1}^{K} EAD_J \times LGD_J \times (ES_\alpha(L_J) - PD_J)$$

Besides (see remarks on additivity of the risk measure), we can give a semi-explicit form of the "Expected Shortfall" in the different sub-portfolios:

$$ES_\alpha(L_J) = \int_{-\infty}^{q_{1-\alpha}(\Theta)} \Phi\left(\frac{\Phi^{-1}(PD_J) - \sqrt{\rho_J}u}{\sqrt{1-\rho_J}}\right)\varphi(u)\,du,$$

where φ represents the Gaussian density and the integral can be quickly computed, by means of a Gaussian quadrature for example.

III Case study

Purpose

The subsequent case study highlights the following points:

(i) Impact of the systemic correlation on Value at Risk and Expected Shortfall.

(ii) Risk contributions in the extended model.

(iii) The difference between risk contributions based on total losses and based on unexpected losses.

Portfolio structure

The portfolio structure is typical of retail banking (see *chart 1*) [17] and comes from a segmentation done in compliance with the Basel II requirements. It is composed of $K = 14$ credit lines. Correlations are obtained through the QIS 3.0 formulas [18].

Credit line	EAD_J	PD_J	LGD_J	ρ_J
1	14%	0,06%	60%	16,7%
2	20%	0,18%	60%	16,1%
3	7%	0,24%	60%	15,8%
4	10%	0,42%	60%	14,9%
5	10%	0,60%	60%	14,2%
6	7%	0,84%	60%	13,2%
7	8%	1,44%	60%	11,1%
8	2%	3,18%	60%	6,9%
9	6%	3,24%	60%	6,8%
10	1%	4,56%	60%	5,0%
11	1%	7,20%	60%	3,2%
12	5%	7,33%	60%	3,2%
13	7%	16,0%	60%	2,1%
14	3%	55,0%	60%	2,0%

Chart 1: **portfolio structure**

Aggregated credit risk

The two studied cases are the regulatory single factor one and the multifactor model with a systemic correlation ρ equal to 50%. In the latter case, an analytical formulation is not available, hence we proceed by Monte Carlo simulation. The chosen confidence level α is 99,9%, equal to the regulatory one. *Chart 2* displays the effect of credit risk diversification and the impact of the choice of the risk measure. Not surprisingly, we find a substantial reduction of VaR and Expected Shortfall by diversification, the magnitude being around 25% here [19]. On the other hand, the use of the risk measure built from the "Expected Shortfall" induces a moderated increase in the credit risk measure, of order 10%.

	ζ (VaR)	κ (Expected Shortfall)
$\rho = 100\%$ (Basel II)	6,1%	6,9%
$\rho = 50\%$ (multifactor model)	4,6%	5,0%
Relative variation	-25%	-27%

Chart 2: **VaR and Expected Shortfall of the aggregated portfolio**

Contributions to aggregated risk

In the single factor case, we have seen that the regulatory risk measure was additive. This is not valid in the general setting that we consider. Still, we have positive homogeneity of order 1 with respect to the exposures on sub-portfolios $EAD_1, \dots, EAD_K$. By writing Euler's equality, we obtain:

$$\zeta = \sum_{J=1}^{K} EAD_J \times \frac{\partial \zeta}{\partial EAD_J},$$

which gives a local decomposition of the measure of the aggregated risk, with respect to exposures at default. The term $EAD_J \times \frac{\partial \zeta}{\partial EAD_J}$ should be understood as the risk contribution of sub-portfolio J to the overall aggregated risk. There exists multiple ways to estimate $\frac{\partial \zeta}{\partial EAD_J}$. It is for example possible to make use of the result proved by Gouriéroux et al. (2000) or by Tasche (2000) which link the partial derivatives of the VaR with respect to allocation, to conditional expectation of losses on the sub-portfolios (in the case of distributions with densities, which applies here):

$$\frac{\partial \zeta}{\partial EAD_J} = E[L_J | L = q_\alpha(L)] - LGD_J \times PD_J,$$

where $L_J = EAD_J \times LGD_J \times \Phi\left(\frac{\Phi^{-1}(PD_J) - \sqrt{\rho_J}\Psi_J}{\sqrt{1-\rho_J}}\right)$ is the loss J. The conditional expectation can be computed by Monte Carlo simulation of the loss distribution, or by non linear regression. Figure 1 represents risk contributions of the sub-portfolios, $EAD_J \times \frac{\partial \zeta}{\partial EAD_J}$, for $J = 1, \dots, 14$ to overall risk for the Basel II case ($\rho = 100\%$) and for the multifactor model ($\rho = 50\%$). In both cases, the credit lines risk contribution to the overall portfolio is the highest for poor quality lines, even if their relative amounts are small. We have here assumed that the quantile based risk measure was based on total losses, rather than unexpected losses. As will be seen below, the contribution of poor quality lines to overall risk comes from their high expected losses.

The risk measure built from the expected shortfall, κ, is positively homogeneous of order 1 with respect to the exposures $EAD_1, \dots, EAD_K$, as well. As a consequence, we can write: $\kappa = \sum_{J=1}^{K} EAD_J \times \frac{\partial \kappa}{\partial EAD_J}$. To estimate $\frac{\partial \kappa}{\partial EAD_J}$, *fig. 1:* **Risk contributions in the aggregated portfolio, VaR case**

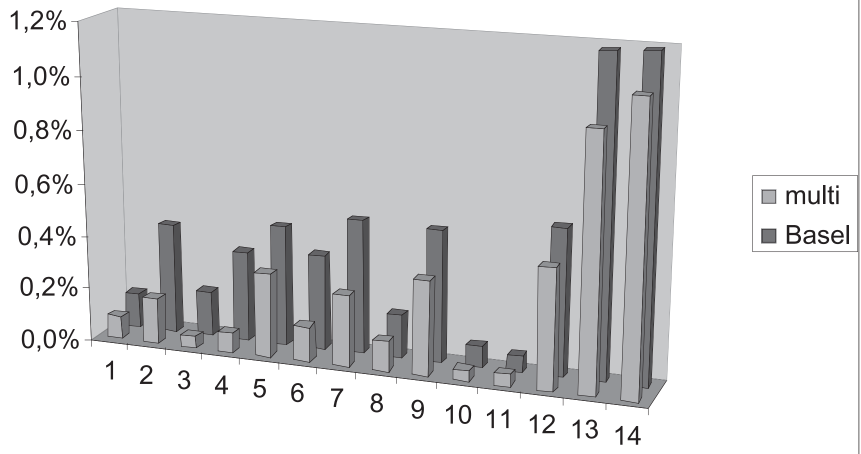
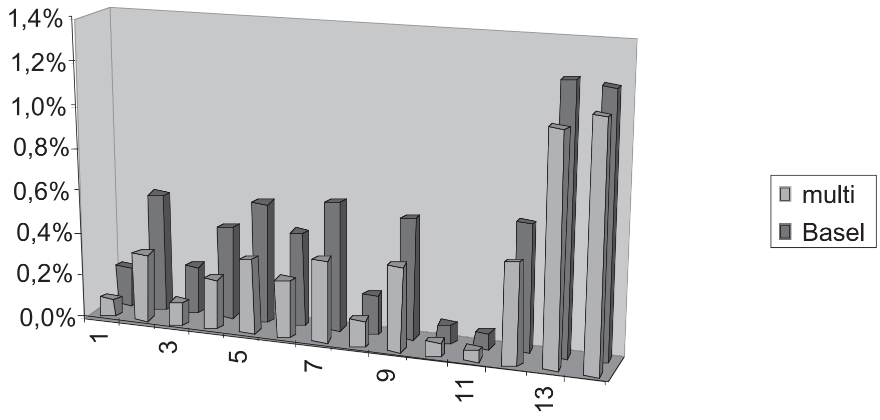


fig. 2: **Risk contributions in the aggregated portfolio, Expected Shortfall case**



we can use the result established by Scaillet (2003) which relates partial derivatives of the "Expected Shortfall" and conditional expectation of losses in the sub-portfolios (in the case of distribution having densities, which applies here):

$$\frac{\partial\kappa}{\partial EAD_J} = E[L_J | L \geq q_\alpha(L)] - LGD_J \times PD_J .$$

The conditional expectation may be estimated by econometric means. We refer to Scaillet (2003) for further discussion on the different approaches.

Figure 2 presents the risk contributions of the sub-portfolios $EAD_J \times \frac{\partial\kappa}{\partial EAD_J}$ when the risk measure is based on the Expected Shortfall [20]. Compared to results obtained with the VaR based risk measure, the overall pattern does not change very much, with a high contribution of poor quality lines $J = 13,14$ though these two lines account for only 10% of total credit exposure. The similarity between the risk contributions when using VaR and ES is due to the smooth pattern of default distributions in the retail banking case. This practical case study also shows that the theoretical critics towards the VaR regulatory measure are somehow artificial in the context of retail credit risk.

Chart 3 represents the relative contributions of total VaR, $\frac{EAD_J}{\zeta} \times \frac{\partial\zeta}{\partial EAD_J}$, for $J = 1, \ldots, 14$ in the case of Basel II ($\rho = 100\%$) and in our multifactor approach ($\rho = 50\%$).

A better account of the diversification effects has quite significant consequences on capital allocation with the multifactor model. The poor quality lines 13 and 14 have then higher relative contributions, while good quality credit lines

Credit line	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Basel II	2,1%	6,8%	2,8%	5,6%	7,4%	5,9%	8,3%	2,7%	8,2%	1,3%	1,0%	9,0%	19,4%	19,5%
multi	2,2%	4,3%	1,0%	1,7%	7,7%	3,2%	6,5%	2,7%	8,6%	1,0%	1,1%	11,0%	23,0%	26,0%

Chart 3: **relative contributions to total VaR**

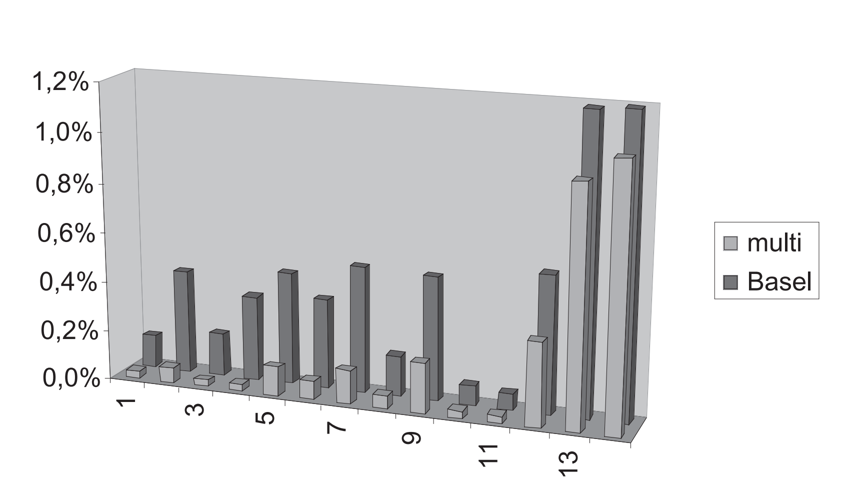


fig. 3: **risk contributions in the aggregated portfolio, the VaR case**

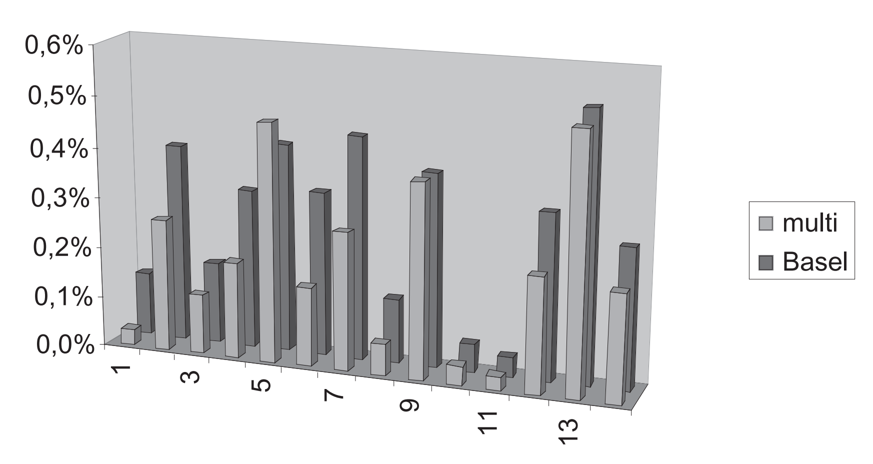


fig. 4: **risk contributions in the VaR case with a recenterered distribution**

have lower relative contributions. This effect is even more apparent when looking for lower systemic correlation.

Figure 3 reports the risk contributions when the systemic correlation ρ is equal to 5%. We can see that almost all the risk is concentrated on poor quality lines $J = 12,13,14$. Clearly, good quality lines benefit from the diversification effect. The reason of such results is mainly due to expected losses effects (see below).

Let us recall that the first three figures and chart 3 are based on total losses rather than unexpected losses. The expected losses do not depend on any correlation parameter and are quite large for poor quality lines. For instance, line 14 is associated with a probability of default equal to 55%. Diversification of credit risk amongst different lines then reduces only unexpected loss. This is enlightened in figure 4. We have here represented relative risk contributions of the sub-portfolios with the VaR risk measure and with the Expected shortfall risk measure, when the measurement is based only on unexpected losses. In both cases, the systemic correlation is set to 50%. We can see a dramatic change in the patterns. Risk is no longer concentrated on poor credit quality lines. One might argue that expected losses cannot be reasonably covered by credit margins for instance in the case of line 14, due to the extremely high probability of default. Nevertheless, it is likely that credit reserves have already been put in front of such highly risky lines.

Consequences on risk management and securitization

In term of risk management, the preceding results help us to define and quantify a strategy of risk reduction in the aggregated portfolio. The purpose of securitization in this context is to create customized portfolio profiles held by the bank, corresponding to the strategy of the Managing Board or to comply with ratings requirements or targets. For example, it is possible to reshape the portfolio by securitizing the line 14 (see *fig. 2*), which nominal amount is around 3% but which concentrates the most part of the risk contribution. In order to decrease the risk of the Special Purpose Vehicle, we can add in the pool some credits from the less risky lines (1 to 9). We can contemplate in this framework a partial cession of the tranches, in order to take into account the case in which the bank retains the first losses. Let us point the fact that even in the case of a partial cession of credit risk [21], the regulatory treatment proposed by the Basel Committee does not agree with an economic approach. Of course, as is clear from our previous discussion, this relies on the assumption that expected losses contribute to the risk of different lines.

Model risk

As a way to assess the impact of parameter choice or estimation, we represent firstly the influence of systemic correlation on the regulatory risk measure ζ and the one built from the Expected Shortfall [22]. We notice an almost linear increasing relationship between the measures and ρ (see *figure 5*).

We display as well the elasticity of the risk measures with respect to the specific correlation parameters: $\frac{\rho_J}{\zeta} \times \frac{\partial \zeta}{\partial \rho_J}$ and $\frac{\rho_J}{\kappa} \times \frac{\partial \kappa}{\partial \rho_J}$ for $j = 1, \ldots, K$ (see *figures 6 and 7*).

A careful risk manager should focus his attention especially on those most sensitive lines, either by testing other credit risk models or by deciding to allocate more capital to those lines. This may be viewed as a conservative attitude towards *uncertainty* on parameters. Picking up the “uncertain” credit lines could be a new criterium to securitize particular assets of the balance sheet. This can be understood as a way of hedging correlation risk, as in financial markets. Moreover, correlations follow the economic cycles. Hence, a wise management of this risk must allow us to tackle and reduce both procyclicality and volatility of capital charges, two major drawbacks of the models studied here. Sensitivity analysis with respect to other parameters is made in the same fashion [23].

Summary

Basel II approach for credit risk does not fully account for portfolio diversification (both geographical and among sectors) among different credit lines. In our case study, this conservative approach leads to an overestimation of capital of an order of magnitude of 25%. On the other hand, using “Expected Shortfall” instead of VaR as a risk measure magnifies economic capital by approximately 10%. As far as retail banking is concerned, the capital allocation using VaR or Expected Shortfall provides similar risk *profiles* be it in the Basel II setup or in our proposed extension. This is due to smooth patterns of loss distributions. Taking into account the diversification of credit risk magnifies the risk contributions of poor quality credit lines by comparison with good quality ones. This appears to be mainly an expected loss effect. Risk measures based on unexpected losses lead to dramatically different capital allocations. ■

fig. 5: **VaR and ES as a function of systemic correlation**

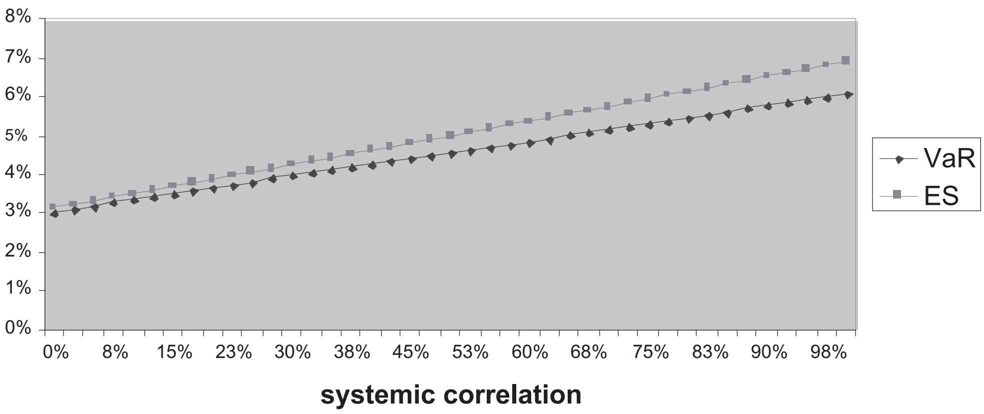


fig. 6: **VaR sensitivity to a one 1% error on correlation**

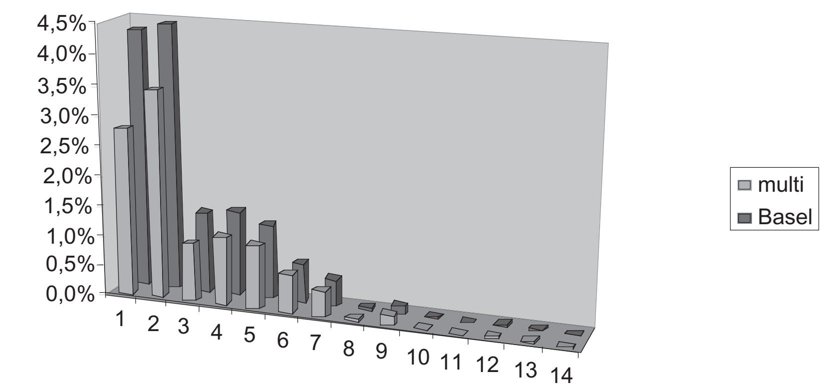
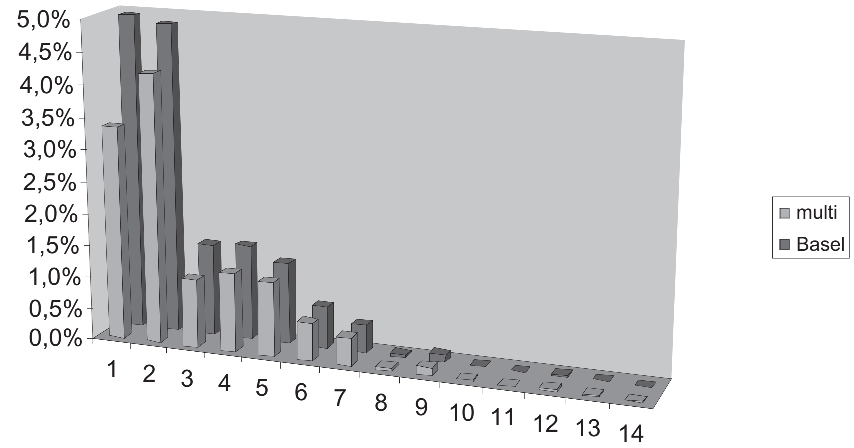


fig. 7: **Expected Shortfall sensitivity to a one 1% error on correlation**



1 Regulatory materials are available on the IRB's website. For the reader interested by a presentation of the models underlying the IRB approach, we refer to Gordy (2001) and Roncalli (2001).
2 On a technical point of view, CreditMetrics and Basel II model are multivariate probit models.
3 "Correlation" must be understood as dependence, not as linear correlation, as the risks we consider here are non Gaussian.
4 However, we show later that it is not restrictive for homogeneous credit portfolios.
5 The Expected Shortfall is a coherent risk measure in the sense of Artzner et al. It enjoys thus desirable properties like sub-additivity, which is not the case of the VaR.
6 Let us remark that the default probabilities are set according to internal ratings, even though these ratings can use information coming from rating agencies.
7 We notice that homogeneity is related to symmetry in the joint law of default indicators.
8 See Wilde (2001), and Dembo et al. (2002).
9 See Düllmann and Scheule (2003), Gordy and Heitfield (2002).
10 It is a translation invariance property which is shared by *coherent risk measures* (see Artzner et al., 1999).
11 For mortgages, the capital charges are given by:

$$\zeta = q_\alpha\left(\sum_{J=1}^{K} EAD_J \times LGD_J \times \Phi\left(\frac{\Phi^{-1}(PD_J) - \sqrt{\rho_J}\Psi_J}{\sqrt{1-\rho_J}}\right)\right).$$

When considering corporate credits, there are other adjustments to cope with finite granularity or maturity effects.
12 It is an easy task to check risk additivity in the Committee's approach. We denote by

$$L_J(\Theta) = EAD_J \times LGD_J \times \Phi\left(\frac{\Phi^{-1}(PD_J) - \sqrt{\rho_J}\Theta}{\sqrt{1-\rho_J}}\right),$$

the loss on credit portfolio J and by $L(\Theta) = \sum_{J=1}^{K} L_J(\Theta)$ the aggregated loss. We notice that the losses $L(\Theta)$, $L_J(\Theta)$ are continuous functions strictly decreasing in Θ. Hence, $-\Theta \le -t \Leftrightarrow L(\Theta) \Leftrightarrow L(t) \Leftrightarrow L_J(\Theta) \le L_J(t)$, $\forall J = 1, \ldots, K$. It implies that $P(-\Theta \le -t) \ge \alpha \Leftrightarrow P(L(\Theta) \le L(t)) \ge \alpha \Leftrightarrow P(L_J(\Theta) \le L_J(t)) \ge \alpha$, $\forall J = 1, \ldots, K$. Looking back at the definition of the lower quantile of order α of a random variable X, $q_\alpha(X) = \inf\{t, P(X \le t) \ge \alpha\}$, and by using again continuity and the one to one relationship between L and L_J, we establish the following equalities: $q_\alpha(L(\Theta)) = L(-q_\alpha(-\Theta))$ and $q_\alpha(L_J(\Theta)) = L_J(-q_\alpha(-\Theta))$,

$\forall J = 1, \ldots, K$. This implies that $q_\alpha(L(\Theta)) = \sum_{J=1}^{K} q_\alpha(L_J(\Theta))$, which was to be demonstrated. By making use of the symmetry of Θ, $-q_\alpha(-\Theta) = q_{1-\alpha}(\Theta)$. This gives a simple expression of the regulatory risk measure in the single factor case:

$$\zeta = \sum_{J=1}^{K} EAD_J \times LGD_J \times \left(\Phi\left(\frac{\Phi^{-1}(PD_J) - \sqrt{\rho_J} q_{1-\alpha}(\Theta)}{\sqrt{1-\rho_J}}\right) - PD_J\right),$$

in which there are only quantiles from the standard Gaussian distribution.

13 We can as well perform a non parametric kernel estimation, see Scaillet (2003).

14 Similarly to the VaR case, for mortgages we should consider a risk measure based on total losses L rather than on unexpected losses $L - E^P[L]$.

15 In particular it is subadditive, whereas the regulatory measure is not (see Artzner et al.(1997, 1999)).

16 It is easy to prove additivity in the single factor setting. We notice that in the case where X has a density we can write $ES_\alpha(X) = E^P[X | X \geq q_\alpha(X)]$. Keeping the same notations as supra where L and L_J are the aggregated loss and the loss on portfolio J, we can write:

$$ES_\alpha(L) = E^P\left[\sum_{J=1}^{K} L_J | L \geq q_\alpha(L)\right] = \sum_{J=1}^{K} E^P[L | L \geq q_\alpha(L)],$$

thanks to linearity of conditional expectation. Similarly to the proof of additivity in the Basel case, we get: $L \geq q_\alpha(L) \Leftrightarrow L_J \geq q_\alpha(L_J)$, $\forall J = 1, \ldots, K$. It follows that:

$$E^P\left[\sum_{J=1}^{K} L_J | L \geq q_\alpha(L)\right] = E^P\left[\sum_{J=1}^{K} L_J | L_J \geq q_\alpha(L_J)\right] = ES_\alpha(L_J),$$

and the additivity of the risk measure built from the Expected Shortfall, in the single factor case.

17 We can notice that lines 2 and 3 or lines 8 and 9 have very similar probabilities of default and losses given default. However, in the multifactor model, one cannot aggregate these lines, since the inter-correlation of default events is different from the intra-correlation of default events.

18 Further information about the Quantitative Impact Study 3.0 can be found at the following webpage: http://www.bis.org/bcbs/qis/qis3.htm

19 It is possible to prove that diversification reduces the risk measure built from the "Expected Shortfall". Denote by $L_J(\Psi_J)$ the loss on portfolio J, Ψ_J being the corresponding risk factor. The aggregated loss can be written $\sum_{K=1}^{J} L_J(\Psi_J)$. Thanks to subadditivity of the risk measure, we obtain that $\kappa\left(\sum_{K=1}^{J} L_J(\Psi_J)\right) \leq \sum_{J=1}^{K} \kappa(L_J(\Psi_J))$. Making use of law invariance of the risk measure, the expression on the right can be rewritten as $\sum_{J=1}^{K} \kappa(L_J(\Theta))$ where Θ is a standard Gaussian random variable. With comonotonic additivity, this is equal to $\kappa\left(\sum_{J=1}^{K} L_J(\Theta)\right)$, which is nothing less than the risk measure corresponding to single factor case.

20 As the for the quantile based risk measure, the expected shortfall based risk measure in figure 2 accounts for total losses rather than unexpected losses. The high contribution of poor quality credit lines is then mainly due to expected losses which are especially high for lines $J = 13, 14$.

21 For instance, the bank sells mezzanine and senior tranches and keeps the equity tranche.

22 We do not discuss the estimation of the ρ parameter (systemic correlation), neither confidence intervals which should be used for estimators of ρ, ρ_J. The estimation process depends on available data which differ from bank to bank.

23 It is available on request.

References

C. Acerbi, "Spectral Measures of Risk: a Coherent Representation of Subjective Risk Aversion," *working paper*, Abaxbank, March 2002.

P. Artzner, F. Delbaen, J-M. Eber, D. Heath, "Thinking Coherently," *RISK*, 10, no. 11, 1997.

P. Artzner, F. Delbaen, J-M. Eber, D. Heath, "Coherent Measures of Risk," *Mathematical Finance*, 9 (3), 203-228, 1999.

Basel Committee on Banking Supervision, "The Internal Ratings-based Approach," *Consultative Document*, Supportive Document to the New Basel Capital Accord, January 2001. See www.bis.org for other documents and quantitative studies.

G. Chamberlain, M. Rothschild, "Arbitrage and Mean Variance Analysis on Large Asset Markets", Econometrica 51, 1281-1304, 1983.

J. Dhaene, S. Vanduffel, Q. H. Tang, M. Goovaerts, R. Kass, D. Vyncke, "Risk measures and Comonotonicity", *working paper*, 2003.

A. Dembo, J-D. Deuschel, D. Duffie, "Large Portfolio Losses," Stanford University, *working paper*, 2002.

K. Düllmann, H. Scheule, "Determinants of asset correlations of German corporations and implications for regulatory capital," Deutsche Bundesbank, working paper, October 2003.

R. Frey, A. McNeil, "Modelling dependent defaults," University of Zürich, *working paper*, 2001.

M. Gordy, "A comparative anatomy of credit risk models," *Journal of Banking and Finance*, 24 (1/2), January 2000, pp. 119-145.

M. Gordy, "A risk-factor foundation for ratings-based bank capital rules," Division of Research and Statistics, Board of Governors of the Federal System, *working paper*, 2002.

M. Gordy, E. Heitfield, "Estimating default correlations from short panels of credit rating performance data," Division of Research and Statistics, Federal Reserve Board, *working paper*, 2002.

C. Gouriéroux, J.-P. Laurent, O. Scaillet, "Sensitivity Analysis of Value at Risk," *Journal of Empirical Finance*, Vol. 7 (2000), 3-4, 225-245.

C. Gouriéroux, C. Monfort, "Equidependence in qualitative and duration models with application to credit risk," CREST, *working paper*, 2002.

T. Roncalli, "Introduction à la Gestion des Risques", 3rd year ENSAI course, 2001.

O. Scaillet, "Nonparametric estimation and sensitivity analysis of expected shortfall," *working paper*, forthcoming in *Mathematical Finance*, 2003.

D.Tasche, "Risk Contributions and Performance Measurement," Technische Universität München, *working paper*, 1999.

D. Tasche, "Conditional Expectation as Quantile Derivative," Technische Universität München, *working paper*, 2000.

D. Tasche, "Expected Shortfall and Beyond," *Journal of Banking and Finance*, 26 (7), 1519-1533, 2002.

O. Vasicek, " The loan loss distribution," KMV corporation, *working paper*, 1997.

T. Wilde, "Probing granularity," *RISK*, 14 (2001), no. 8, 103-106.

Appendix: convergences for infinitely granular portfolios

We provide here a precise meaning to both notions of "infinitely granular portfolio" used in the regulatory texts and credit risk diversification. In the sequel $(\Omega, \Im, P)$ is a space endowed with a probability measure, Ψ refers to a random vector of dimension $d \geq 1$ and L_k, $k \in \mathbb{N}$ are random variables representing losses on credits. The variables L_k, $k \in \mathbb{N}$ are supposed to be conditionally independent knowing Ψ. Besides, we will assume that $0 \leq L_k \leq 1$, $k \in \mathbb{N}$ (losses on a credit are positive and are at most equal to the nominal of the credit). P^{Ψ} is a regular version of a probability measure knowing Ψ. We will suppose that losses L_k are identically distributed under P^{Ψ}. Let us recall that if *X* is a random variable with finite expectation under *P*, then $E^P[X|\Psi] = E^{P^{\Psi}}[X]$. Here losses L_k have finite expectations under *P*, hence under P^{Ψ}. We write $S_n = \frac{1}{n}\sum_{k=1}^{n} L_k$ the aggregated loss for *n* credits. We can then establish the following result:

Property (convergence in distribution for infinitely granular portfolios):

$$S_n = \frac{1}{n}\sum_{k=1}^{n} L_k \xrightarrow{\text{d}} E^P[L_1|\Psi],$$

where $\xrightarrow{\text{d}}$ refers to the convergence in distribution.

Proof: We will use the Lévy theorem. The characteristics function of S_n, φ_{S_n}, defined by $\varphi_{S_n}(u) = E^P[\exp(iuS_n)]$, for $u \in \mathbb{R}$ can be written as:

$$\varphi_{S_n}(u) = E^P\left[E^{P^{\Psi}}\left[\exp\left(iu\frac{1}{n}\sum_{k=1}^{n} L_k\right)\right]\right]$$

by the iterated expectations theorem. Let us denote $z_n = E^{P^{\Psi}}\left[\exp\left(iu\frac{1}{n}\sum_{k=1}^{n} L_k\right)\right]$.

We then have:

$$E^P[z_n] = \varphi_{S_n}(u) = E^P\left[\prod_{k=1}^{n} E^{P^{\Psi}}\left[\exp\left(iu\frac{L_k}{n}\right)\right]\right]$$

$$= E^P\left[\left(E^{P^{\Psi}}\left[\exp\left(iu\frac{L_1}{n}\right)\right]\right)^n\right]$$

where we have used the independence of the L_k conditionally to Ψ and the invariance in distribution of the L_k under P^{Ψ}. Let us do an expansion at 1st order:

$$E^{P^{\Psi}}\left[\exp\left(iu\frac{L_1}{n}\right)\right] = 1 + \frac{iu}{n}E^{P^{\Psi}}[L_1] + o\left(\frac{1}{n}\right)$$

with $\lim_{n \to \infty} no\left(\frac{1}{n}\right) = 0$.

We notice that $z_n = \left(1 + \frac{iu}{n}E^{P^{\Psi}}[L_1] + o\left(\frac{1}{n}\right)\right)^n$, hence: $\lim_{n \to \infty} z_n = \exp(iuE^{\Psi}[L_1])$. Furthermore $|z_n| \leq E^{P^{\Psi}}\left[\left|\exp\left(iu\frac{1}{n}\sum_{k=1}^{n} L_k\right)\right|\right] = 1$. The random variables z_n being less than 1, the dominated convergence theorem leads to: $\lim_{n \to \infty} E^P[z_n] = E^P[\exp(iuE^{P^{\Psi}}[L_1])]$.

The second part of the previous equality is nothing else than $\varphi_{E^{P^{\Psi}}[L_1]}(u)$, the value taken in u of the characteristics function of $E^{P^{\Psi}}[L_1]$.

Hence:

$$\lim_{n \to \infty} \varphi_{S_n}(u) = \varphi_{E^{P^{\Psi}}[L_1]}(u),$$

which shows the convergence in distribution of S_n to $E^{P^{\Psi}}[L_1] = E^P[L_1|\Psi]$.

We can establish the following result as well:

Property (convergence almost surely for infinitely granular portfolios):

$$S_N = \frac{1}{N}\sum_{k=1}^{N} L_k \xrightarrow{P\text{-}as} E^P[L_1|\Psi],$$

where $\xrightarrow{P\text{-}as}$ refers to convergence *P*- almost surely.

Proof: We notice that the random variables L_k, $k = 1, \ldots, K$ are independent under P^{Ψ}. The variables L_k are identically distributed under P^{Ψ}. Furthermore, as $0 \leq L_k \leq 1$, $k \in \Psi$, we have $E^{\mathbb{P}^{\Psi}}(|L_k|) \leq 1$. Thanks to the law of large numbers, we obtain:

$$P^{\Psi}\left(\frac{1}{n}\sum_{k=1}^{n} L_k \to E^{P^{\Psi}}[L_1]\right).$$

Using the general version of Fubini's theorem:

$$P\left(\frac{1}{n}\sum_{k=1}^{n} L_k \to E^{P^{\Psi}}[L_1]\right) = \int_{\Omega} P^{\Psi}\left(\frac{1}{n}\sum_{k=1}^{n} L_k \to E^{P^{\Psi}}[L_1]\right) dP^{\Psi}$$

$$= 1$$

which proves the convergence *P*- almost surely of S_n to $E^{P^{\Psi}}[L_1] = E^P[L_1|\Psi]$. Thus, the loss on an infinitely granular portfolio can be written as a measurable function of the factor Ψ, the specific risks of the credits being "eliminated" by diversification. ■