Libor Market Model in Premia*

Bermudan Pricer, Stochastic Volatility and Malliavin Calculus

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I. Introduction

The aim of Premia is to provide with numerical algorithms applied to the pricing of derivative products with the relevant mathematical details. Up to Premia 6 all the focus was on equity derivative products. All known numerical methods, from partial differential equations methods to Monte Carlo methods were applied to equity derivative pricing and/or calibration. Therefore it was rather natural to start the implementation of the models devoted to other derivatives markets. The interest rate derivatives market appears to be the next challenging step.

Since the seminal work of Vasicek [26] on the pricing of a bond in a stochastic interest rates framework a large number of works dealing with the pricing of interest rate derivatives were proposed by academies. The first extensions of this model made in the early eighties mainly specified different dynamics for the short rate thus following the modeling strategy proposed by Vasicek. The focus was on bond pricing and the pricing of a call/put on a bond and option on a coupon bearing bond which were the most important derivatives in the eighties. One of the main difficulty with this approach was the lack of consistency with the initial term structure of interest rates. As pointed out by Cox, Ingersoll and Ross [10] this problem could be overcome by letting the parameters to be time dependent. Nevertheless this solution proved to be unsatisfactory in certain case. It leads Heath, Jarrow and Morton [14] to reformulate the modeling of interest rates by taking as state variables not only the short rate but all the (instantaneous) forward rate curve thus building a framework which is consistent with the initial term structure by construction. Hull and White [15] proposed a short rate dynamics which is consistent with the initial term structure and allows to compute bond option in closed form. Obviously their model falls within the Heath-Jarrow-Morton framework, see Bjork [3].

Despite all efforts made by academies the market practice was to price a caplet, an option on a simple interest rate, using the Black's formula [4] and was inconsistent with known specification of the Heath-Jarrow-Morton model. It was Brace, Gatarek and Musiela [5] and Miltersen, Sandmann and Sondermann [19] who proposed to take as state variables not the instantaneous forward rates but the simple rates thus leading to a closed form solution for caplet options consistent with practice. This framework was then further completed by Brace, Gatarek and Musiela [5] who clarified the constraints implied by arbitrage freeness on the dynamics of the rates. As the state variables in this model are the observable simple rates this model was called "market model" and is now named "Libor Market Model" (LMM in the sequel). Being widely used in practice it was decided to mainly focus on this model although the main known results for the Gaussian and Cox-Ingersoll-Ross [10] models were also developed in Premia 7. Moreover some of the results presented here are new thus justifying the choice to devote this paper only to the LMM.

In this document we will present different works dealing with the Libor Market Model and implemented in Premia 7. We will focus on the models and results and report to the documentation Barton-Smith *et al.* [2] for any aspects related to the application programming interface. Furthermore some algorithms such as arbitrage free discretization of the LMM and the "Bushy tree" technique for the LMM was not reported to lighten the presentation. Again the interested reader will find in the library a complete description of those algorithms in Barton-Smith *et al.* [2].

The paper is organized as follows: in the next section we present the framework associated with the Libor Market Model, we specify the notations, the variables, the main assumptions that underline this model and one of the most important interest rate derivative product namely the (European) swaption which is used to calibrate the model. In section 3, we present the pricing of

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the Bermudan swaption using Monte Carlo techniques, we emphasize the importance of the numeraire used to price the product and the rôle of the state variables involved in the least squares regression. These results underline the differences with known results on equity derivatives and are new. In section 4, a stochastic volatility extension of the Libor Market Model proposed by Wu and Zhang [25] in order to handle the smile observed in the Cap/Floor/Swaption market. Even if we merely provide an implementation of the model without new research results it is worth to point out that the availability of the code is an important aspect of Premia. In fact it allows the users to check their own implementation thus helping to develop reliable programs but also because not all of the computational details can be put in a paper. In section 5, we show how to compute the sensitivities for an European swaption using the Malliavin calculus. We briefly presents the main known results of this technique when applied to equity derivatives. We explain what is the main dimculty when we try to extend it to interest rates derivatives. Then we provide a solution to this problem and illustrate by Computing the delta and the gamma of a European swaption its emciency. These results are new. Section 6 concludes the paper.

II. Libor Market Model: the framework

We start by a short overview of the Libor market model with discrete tenor structure and refer to the literature for a more in depth presentation Brace, Gatarek and Musiela [5] and Musiela and Rutkowski [20].

We suppose the following date structure $\{T_n = T_0 + n\tau; n = 1, \ldots, M\}$ which is regularly spaced for simplicity. We will note $T_0 = t$ the current time in the sequel. We note $B_t^T = B(t, T)$ the value of a zero coupon bond at time t with maturity T. When T is within the tenor structure we will use the notation $B_t^i = B_t^{T_i}$. The forward Libor rate, noted $L_t^i = L(t, T_i, \tau) = L(t, T_i)$, is given by the well known relation

$$1 + \tau L_t^i = \frac{B_t^i}{B_t^{i+1}} \quad (1)$$

It is the forward rate which sets at time T_i the cash flow received at time $T_i + \tau$. The forward swap rate starting at date T_s and ending at T_M is given by

$$S_t^{s,M} = \frac{B_t^s - B_t^M}{\sum_{j=s+1}^{M} \tau B_t^j} = \frac{1 - \prod_{j=s}^{M-1} \frac{1}{1 + \tau L_t^j}}{\sum_{j=s+1}^{M} \tau \prod_{k=s}^{j-1} \frac{1}{1 + \tau L_t^k}}. \quad (2)$$

The spot swap rate is $S_{T_s}^{s,M}$.

Following Brace, Gatarek and Musiela [5] we suppose for the forward Libor rates a dynamics of the form

$$dL_t^j = L_t^j \gamma_t^j \cdot dW_t^{Q^{j+1}}$$

where $\left\{W_t^{Q^{j+1}}; t \geq 0\right\}$ is a d-dimensional Brownian motion under the forward probability Q^{j+1} associated with the numeraire B_t^{j+1} and $\gamma_t^j = \left(\gamma_t^{j,1}, \ldots, \gamma_t^{j,d}\right)^{*1}$ is a d-dimensional deterministic vector function[2].

It is of interest to specify the dynamics of the Libor L_t^j under the probability Q^s. Using Girsanov's formula we have for $j \in \{s, \ldots, M-1\}$

$$\frac{dL_u^j}{L_u^j} = \gamma_u^j \cdot dW_u^{Q^s} + \sum_{k=s}^{j} \frac{\tau L_u^k \gamma_u^k \cdot \gamma_u^j}{1 + \tau L_u^k} du$$

where $\left\{W_u^{Q^s}; u \geq t\right\}$ is a d-dimensional Brownian motion under Q^s. In a similar way the dynamics of L_t^j under Q^s for $j < s - 1$ is given by

$$\frac{dL_u^j}{L_u^j} = \gamma_u^j \cdot dW_u^{Q^s} - \sum_{k=j+1}^{s-1} \frac{\tau L_u^k \gamma_u^k \cdot \gamma_u^j}{1 + \tau L_u^k} du$$

Using this model it greatly simplifies to freeze the drift as it was first suggested by Brace, Gatarek and Musiela [5] therefore we will state the following dynamic for the forward rates under Q^s for $j \geq s$ and $u \in [t, T_s]$

$$\frac{dL_u^j}{L_u^j} = \gamma_u^j \cdot dW_u^{Q^s} - \sum_{k=s}^{j} \frac{\tau L_t^k \gamma_u^k \cdot \gamma_u^j}{1 + \tau L_t^k} du$$

Under this assumption the forward rates are lognormal and Itô's Lemma leads to

$$\begin{aligned} L_{T_s}^j &= L_t^j e^{\int_t^{T_s} \gamma_u^j \cdot dW_u^{Q^s} - \frac{1}{2}\int_t^{T_s} \gamma_u^j \cdot \gamma_u^j du + \sum_{k=s}^{j} \frac{\tau L_t^k}{1 + \tau L_t^k} \int_t^{T_s} \gamma_u^k \cdot \gamma_u^j du} \\ &= L_t^j e^{\int_t^{T_s} \gamma_u^j \cdot dW_u^{Q^s} - \frac{1}{2}\left(\gamma^j \cdot \gamma^j\right) + F_{T_s}^j} \end{aligned} \quad (3)$$

The same approximation can be done for the dynamics of the Libor rates under the other probabilities.

In this document the focus will be on swaptions, both Bermudan and European, and caplet/floorlet. These products are of interest because of their strong liquidity. For example European swaptions in addition with caps and floors[3] are used to calibrate the Libor Market Model.

The European payer swaption is noted $Swpt(t, T_s, T_M, K, \tau, N)$, where T_s is the maturity, K strike, N nominal which for simplicity we choose equal to 1, and T_M the underlying swap maturity (the tenor of the swaption is $T_M - T_s$). At swaption maturity exercising the option leads the cash flows equal to $N\tau\left(S_{T_s}^{s,M} - K\right)_+$ at times $\{T_i; i = s+1, \ldots, M\}$.

Standard arbitrage arguments give the pricing formula

$$Swpt(t, T_s, T_M, K, \tau) = \sum_{i=s+1}^{M} E_t^Q\left[\frac{B_t}{B_{T_i}} \tau\left(S_{T_s}^{s,M} - K\right)_+\right]$$

where B_t is the cash and Q is the risk neutral probability (associated with the cash as numeraire). For this product it is more convenient to price under the forward measure Q^s linked to Q through Girsanov's theorem

$$
\begin{aligned}
Swpt(t, T_s, T_M, K, \tau) &= B_t^s E_t^{Q^s}\left[\sum_{i=s+1}^{M} B_{T_s}^i \tau\left(S_{T_s}^{s,M} - K\right)_+\right] \\
&= B_t^s E_t^{Q^s}\left[\sum_{i=s+1}^{M} B_{T_s}^i \tau \left(\frac{1 - B_{T_s}^M}{\sum_{j=s+1}^{M} \tau B_{T_s}^j} - K\right)_+\right] \\
&= B_t^s E_t^{Q^s}\left[\left(1 - \sum_{i=s+1}^{M} \widetilde{K}_i B_{T_s}^i\right)_+\right] \\
&= B_t^s E_t^{Q^s}[\Phi(f(.))]
\end{aligned}
$$

with $\{\widetilde{K}_i = \tau K ; i = s, ..., M-1, K_M = 1 + \tau K\}$, $\Phi(x) = (x)_+$ and $f(.)$ is a function of the forward Libor rates:

$$
f\left(\left\{L_{T_s}^k ; k = s, ..., M-1\right\}\right) = 1 - \sum_{k=s+1}^{M} \widetilde{K}_k \prod_{l=s}^{k-1} \frac{1}{1 + \tau L_{T_s}^l} \qquad (4)
$$

III. Bermudan swaption pricing in the Libor Market Model

In preceding section we introduced the most common *vanilla* interest rate product the European Swaption, the aim of the present one is to describe some *exotic* products which undergo the name of Bermudan Swaptions. The main feature of a Bermudan option is the possibility to exercise the option at different exercise dates. In order to price this product one has to solve a dynamical programming problem which involves the computation of a conditional expectation. Carriere [7] proposed an algorithm based on least squares method and Monte-Carlo simulation. It was further applied by Longstaff and Schwartz [17] to the pricing of different derivatives. In Premia 3, Cohort [9] provided an implementation of this algorithm when applied to the pricing of equity derivatives. The conditional expectation, which is a non linear function of the stocks, is approximated by a polynomial whose coefficients are computed through least square regression. The state variables are the stocks and the dynamics is the usual geometrie Brownian motion.

To extend this approach to the pricing of interest rates derivatives one has to specify two things. The first one is the choice of the numeraire. To price equity derivatives one usually takes the risk neutral martingale measure. In the case of interest rates derivatives several pricing measures are available with different numerical efficiency. The second aspect to be specified is the choice of the state variables used *in the regression*. When dealing with equity derivatives the state variables use to be the stocks but when dealing with interest rates derivatives the rates may not be the regression variables. For example in the case of a payer swaption the option is exercised if the swap rate is sumciently above the strike therefore it is tempting to use the swap rate as a regression variable even though the model is specified through the dynamics of the *simple* rates. As a consequence there are the state variables specifying the dynamics of the model and there are the state variables used in the least square regression. They might be different. Therefore the extension of this algorithm to the pricing of interest rates derivatives is not straightforward and we provide examples illustrating those difficulties.

Let us consider as before a tenor structure $T_0, ..., T_M$ and define an *ending date* T_e and a *starting date* T_s such that $T_0 \leq T_s \leq T_e \leq T_M$. There are (at least) two possibility of setting up a Bermudan Swaption agreement: one with *fixed-maturity* and one with *fixed-length*. A fixed-maturity Bermudan Swaption is an agreement which gives the owner the right of choosing, at each T_i with $s \leq i \leq e$, whether to enter or not into a European Swaption over $[T_i, T_e]$, while a Bermudan Swaption with a fixed length of $m \in \mathbb{N}$ tenor periods, will give the owner the right to enter into a European Swaption over $[T_i, T_{i+m}]$. From now on, we will restrict to the case of fixed-maturity Bermudan Swaptions, extension to fixed-lenght ones being straightforward[4].

Entering a payer $[T_i, T_e]$-European Swaption with strike K and nominal value $\overline{N}$, means to be payed at time T_i, the quantity[5]

$$
Swpt(T_i ; T_i, T_e, K, \tau) \doteq \left(\sum_{j=i+1}^{e-1} \tau B_{T_j}^e\right)\left(S_{T_i}^{i,e} - K\right)_+ \overline{N} \qquad (5)
$$

where $S_{T_i}^{i,e}$ is the swap rate corresponding to the chosen swap agreement. Thus, price at time $t < T_s$ of a Bermudan swaption with starting date T_s and fixed-maturity T_e, is given, under the measure Q^Y corresponding to a given price process Y as numeraire, by

$$
U(t) = \sup_{\bar{\tau} \in \mathcal{T}^{s,e-1}} E_t^{Q^Y}\left[\frac{Y(t)}{Y(\bar{\tau})} Swpt(\bar{\tau} ; \bar{\tau}, T_e)\right], \qquad (6)
$$

where $\mathcal{T}^{s,e-1}$ is the set of stopping times $\bar{\tau}$ taking values in $\{T_s, ... T_{e-1}\}$. Standard theory of optimal stopping time Lamberton and Lapeyre [16] ensures us that $U(0)$ is the solution of the following dynamic programming problem:

$$
\begin{cases}
U_{e-1} = Swpt(T_{e-1} ; T_{e-1}, T_e) \\
U_i = \max\left\{Swpt(T_i ; T_i, T_e), E_{T_i}^{Q^Y}\left[\frac{Y(T_i)}{Y(T_{i+1})} U_{i+1}\right]\right\}, \\
\qquad \forall i = s, ... e-2 \\
U_0 = E_{T_0}^Y\left[\frac{U_s}{Y(T_s)}\right]
\end{cases} \qquad (7)
$$

where for simplicity of notation we set $U(T_i) = U_i$. The dynamic programming formulation is clearly less synthetic than the optimal stopping time one but is of easier implementation.

3.1. MONTE CARLO PRICING OF BERMUDAN SWAPTIONS WITH THE LONGSTAFF-SCHWARTZ ALGORITHM

Due to the necessity of comparing at each time T_i the exercise value $Swpt(T_i; T_i, T_e)$ to the continuation value $V_i \doteq E^{Q^Y}_{T_i}\left[\frac{Y(T_i)}{Y(T_{i+1})}U_{i+1}\right]$, derivatives on high dimensional underlying (here the forward rates) is usually performed *via* Monte Carlo simulations or by means of trees methods. In Premia we implement the MC algorithm which was firstly introduced by Longstaff and Schwartz in 2001 and which is based on a least squares approach. As this algorithm is widely described in the PremiaDoc section devoted to Monte Carlo methods for asset derivatives, here we will only report main ideas for self-consistency.

Let us take a look to equation (7): in order to price a Bermudan Swaption, it is clear that we must be able to evaluate continuation values, a task which a priori is nor easy nor straightforward. However, by definition of conditional expectation, each continuation value V_i can be seen as the best L^2-approximation of the discounted T_{i+1} price $Y(T_i)U_{i+1}/Y(T_{i+1})$ among the $\mathcal{F}_{T_i}$-measurable random variables. Thus, following the authors, we choose an $\mathcal{F}$-adapted and square integrable stochastic process $X(t)$ together with a set of basis functions $\underline{e} = (e_1, \ldots, e_m)$ such that $E^{Q^Y}\left[e_i^2(X(t))\right] < \infty$ for all $t \leq T_e$ and we set

$$V_i \approx \underline{a}_i \cdot \underline{e}(X(T_i)) \tag{8}$$
$$\underline{a}_i = \arg\min_{\underline{a} \in \mathbb{R}^M} E^{Q^Y}\left[\frac{Y(T_i)}{Y(T_{i+1})}U_{i+1} - \underline{a} \cdot \underline{e}(X(T_i))\right]^2.$$

In other words, for all $i = s, \ldots, e-1$ we find the best approximation of

$$Y(T_i)U_{i+1}/Y(T_{i+1})$$

in the m-dimensional subset of L^2 spanned by $\{e_1(X(T_i)), \ldots, e_m(X(T_i))\}$. The goodness of such an approximation will rely on the "*explanatory power*" of X and on the choice of the basis functions. The Longstaff and Schwartz algorithm is then based on finding an approximated solution to the least squares problem (8) by considering N independent samples of the forward rates stochastic process $L_t = \{L_t^0, \ldots L_t^{e-1}\}$ and of the *explanatory variable* $X(t)$. It is then natural to approximate regression coefficients $\underline{a}_i$ by

$$\underline{a}_i \approx \underline{a}_i^N = \tag{9}$$
$$\arg\min_{\underline{a} \in \mathbb{R}^M} \frac{1}{N}\sum_{n=1}^{N}\left[\frac{B^{(n)}(T_i)}{B^{(n)}(T_{i+1})}U_{i+1}^{(n)} - \underline{a} \cdot \underline{e}(X^{(n)}(T_i))\right]^2$$

where the superscript $^{(n)}$ stand for the n-th Monte Carlo call. Finally, the dynamic programming problem (7) rewrites,

$$\begin{cases} U_{e-1}^{N,(n)} = Swpt^{(n)}(T_{e-1}; T_{e-1}, T_e) \\ U_i^{N,(n)} = \max\{Swpt^{(n)}(T_i; T_i, T_e), \underline{a}_i^N \cdot \underline{e}(X^{(n)}(T_i))\}, \\ \quad \forall i = s, \ldots, e-2 \\ U_0^N = \frac{1}{N}\sum_{n=1}^{N}\left[\frac{U_s^{(n)}}{B^{(n)}(T_s)}\right] \end{cases} \tag{10}$$

Clement, Lamberton and Protter [8] have proved convergence of such an algorithm to the original problem when N and m go to $+\infty$. In particular, when $N \to \infty$, a central limit theorem holds.

3.2. PREMIA IMPLEMENTATION

Implementing the Longstaff and Schwartz algorithm, requires the choice of a set of basis functions and of an explanatory variable.

Concerning the basis functions, Premia algorithm allows for two options: a canonical polynomial basis and an Hermite polynomial basis. At each timestep T_i, optimal coefficients $\underline{a}_i^N$, are found by regressing the $B^{(n)}(T_i)U_{i+1}^{(n)}/B^{(n)}(T_{i+1})$ over the $X^{(n)}(T_i)$. Moreover, for early steps of backward dynamic programming (regression for $T_{e-2}, T_{e-3}, \ldots$) the price will not be very different from exercise value. With our algorithm it is thus possible to include the early exercise value in the regression basis, that is to set for the first basis function $e_1(X^{(n)}(T_i)) = Swpt^{(n)}(T_i; T_i, T_e)$ for all i greater than a given $\bar{i}$.

On the other hand, the choice of an explicatory variable is more tricky and constrained by the necessity of keeping m quite small in order to make regression fast. Let us recall that the swap rate $S_{T_i}^{i,e}$ is indeed a function of $(L_{T_i}^i, \ldots, L_{T_i}^{e-1}$ and then the more natural explanatory variable would be the Libor state vector L_t. However, consider a Bermudan swaptions with $\tau = 0.5$ years, $T_s = 3.0$ yars, $T_e = 8.0$ years. Following the above reasoning, we would need two Libors for regressing at time T_{e-2} but we would need nine Libors to regress at time T_s. Thus, for maturities close to T_s we must have $m \approx 10$ to take into account all relevant Libors. Whenever considering long maturity swaptions, things get even worse. That is why Pieterz *et al.* [23] suggest to regress directly on the Notional Paying Value (NPV) $Swpt(\cdot, \cdot, T_e)$ while Pedersen [22] do test regression on the numeraire, the fixed leg value $K\left(\sum_{j=i+1}^{e-1}\overline{N}\tau B_{T_j}^e\right)$ and the prices of some European options embedded in the Bermudan contract. Pedersen concludes that the European prices are not relevant and that a quadratic function in the numeraire and in the fixed leg value is accurate enough.

Premia users can set the explanatory variable to be

- the notional paying value
- the underlying Brownian motion
- the numeraire.

In table 3.2 we report Premia pricing results and compare them to the ones obtained by Pietersz. *et al.* [23] with a Longstaff and Schwartz algorithm and their drift approximation method. In particular, we take a 1-Factor model with flat volatility (15%) and initial forward rate values (5%); the tenor τ is 0.5 years and the SDE is discretized with 10 timesteps for each tenor period. We price At-The-Money Bermudan swaptions for various choices of starting and ending times T_s and T_e and changing the explanatory variable (Brownian, NPV, numeraire). Regression basis is four dimensional ($m = 4$) and we used 10000 Monte Carlo calls.

Remark: We strongly recommend to include NPV in regression basis when regressing onto the Brownian motion or the numeraire, especially for short length swaption, in which the difference between the European and the Bermudan contract is small. On the other side, when regressing on the NPV, take care NOT to include the payoff into regression[6] because it is likely to waste the performance of regression (Cholesky routine used for regression could return errors). For instance, regressing on the numeraire, a choice $T_{\bar{i}} = 5$ years is good enough either for a swaption with $(T_s, T_e) = (5{,}8)$ and for a (1,8) swaption. When the length of the longest swaption (T_s, T_e) is short, regression on Brownian motion seems to be more stable with respect to changes in $T_{\bar{i}}$ than regression on the numeraire.

In conclusion for interest rate derivatives models a special care is necessary. The state variables used in the regression are the state variables defining the model are not necessarily the same. Furthermore the notion of numeraire is much more important for pricing in the interest rate derivatives world than in the equity derivatives world. Those two aspects will have a strong impact on this algorithm and this study precisely sheds light on this fact, this is the main contribution of this part. Nevertheless the convenient choice of the numeraire and the state variables is more an art than a science.

IV. Libor market model: a stochastic volatility extension

The market of OTM and ITM cap/floor is now liquid and as in the equity world we observe a smile. This leads to the development of smiled models. One of them was proposed by Wu and Zhang in [25] and was implemented in Premia 7.

The authors allows the volatility of the Libors to be stochastic using a square root process as in the Heston model [18]. In that case only the Fourier transform of the density is known in closed form and the pricing is done through Fast Fourier Transform as in Carr and Madan [6] or by numerical integration of the characteristic function. In that case it is merely an implementation and unfortunately we were not able to test on real data this model. The calibration exercise has been carried out in a subsequent release of Premia (see Privault and Wei [24]).

We recall the main equations of the article of Wu and Zhang [25] within the notation of this document.

4.1. THE MODEL

Under the risk neutral measure Q the zero coupon bond follows the dynamic

$$\begin{cases} \dfrac{dB_t^T}{B_t^T} &= r(t)dt + \sqrt{V_t}\,\sigma_B(t,T)\cdot dW_t^Q \\ dV_t &= \kappa(\theta - V_t)dt + \varepsilon\sqrt{V_t}\,dZ_t^Q \end{cases}$$

Table 1. Fixed-maturity Bermudan Swaptions prices for different starting and ending time

	Pietersz *et al.*			Premia		
$[T_s,T_e]$	Drift Approx	L-S	Std Err	Brown.	numer.	NPV
1,2	29.40	28.85	0.42	29,36	28,91	29,35
1,3	64.33	62.78	0.83	64,66	63,61	64,75
1,4	101.66	101.51	1.29	102,98	102,36	103,14
3,4	44.09	43.59	0.70	43,74	43,70	43,74
1,6	182.16	179.48	2.22	185.65	184,59	185,67
3.6	134.88	136.43	2.01	134,84	132,40	134.96
5,6	50.93	50.79	0.86	49,66	49,65	49,66
1,8	266.63	266.35	3.15	264,00	262,11	263,83
3,8	226.55	226.94	3.14	223.85	219,54	223,94
5,8	151.23	151.13	2.38	148.07	148.10	148,13
7,8	54.20	53.70	0.96	52.50	52.48	52,49

Pietersz, Pelsser and von Mortengel [23] Drift Approximation and Longstaff-Schwartz 1 Factor results compared to Premia 1-factor Longstaff-Schwartz with different choices for the explanatory variable.

where $(W_t; t \geq 0)$ is a d dimensional Brownian motion under Q, $(Z_t; t \geq 0)$ is a 1 dimensional Brownian motion under Q, and $\sigma_B(t, T)$ is a d-dimensional vector. For the Libor rates we have

$$\frac{dL_t^j}{L_t^j} = \sqrt{V_t}\gamma_t^j \cdot \left[dW_t^Q - \sqrt{V_t}\sigma_B(t, T_{j+1})dt\right] \quad (11)$$

$$dV_t = \kappa(\theta - V_t)dt + \varepsilon\sqrt{V_t}dZ_T^Q \quad (12)$$

with

$$\gamma_t^j = \frac{1 + \tau L_t^j}{\tau L_t^j}\left[\sigma_B(t, T_j) - \sigma_B(t, T_{j+1})\right]. \quad (13)$$

In the Libor market model we make the hypothesis that $\gamma_t^j = \left(\gamma_t^{j,1}, \ldots, \gamma_t^{j,d}\right)^*$ are deterministic functions. From (13) and under the hypothesis $\sigma_B(t, T_1) = 0$ we obtain

$$\sigma_B(t, T_{j+1}) = -\sum_{k=1}^{j} \frac{1 + \tau L_t^j}{\tau L_t^j}\gamma_t^\kappa.$$

The correlation between the forward rate factors and the volatility factor is given by

$$\frac{\gamma_t^j \cdot dW_t^Q}{\|\gamma_t^j\|}dZ_t^Q = \rho_j(t)dt.$$

If we note $W_t^Q = \left(W_t^{1,Q}, \ldots, W_t^{d,Q}\right)$ and $dW_t^{i,Q}dZ_t^Q = \rho^i dt$ we have $\|\gamma_t^j\|\rho_j(t)dt = \gamma_t^j \cdot dW_t^Q dZ_t^Q = \sum_{i=1}^{d}\rho^i\gamma_t^{i,j}dt$. Under Q^{j+1} the probability measure associated with B_t^{j+1} as numeraire we have

$$\begin{cases} \dfrac{dL_t^j}{L_t^j} = \sqrt{V_t}\gamma_t^j \cdot dW_t^{Qj+1} \\ dV_t = \kappa\left(\theta - \left(1 + \frac{\varepsilon}{\kappa}\xi_j(t)\right)V_t\right)dt + \varepsilon\sqrt{V_t}dZ_t^{Qj+1} \end{cases}$$

where $\left(W_t^{Qj+1}; t \geq 0\right)$ resp. $\left(Z_t^{Qj+1}; t \geq 0\right)$ is a d resp. 1 dimensional Brownian motion under Q^{j+1} and

$$\xi_j(t) = \sum_{k=1}^{j} \frac{\tau L_t^k}{1 + \tau L_t^k}\rho_k(t)\|\gamma_t^k\|.$$

The authors propose to freeze this stochastic process and define

$$\xi_j^0(t) = \sum_{k=1}^{j} \frac{\tau L_0^k}{1 + \tau L_0^k}\rho_\kappa(t)\|\gamma_t^k\|.$$

Using this function we write $\tilde{\xi}_j(t) = 1 + \frac{\varepsilon}{\kappa}\xi_j(t)$ and $\tilde{\xi}_j^0(t) = 1 + \frac{\varepsilon}{\kappa}\xi_j^0(t)$ thus the dynamics is given by

$$\begin{cases} \dfrac{dL_t^j}{L_t^j} = \sqrt{V_t}\gamma_t^j \cdot dW_t^{Qj+1} \\ dV_t = \kappa\left(\theta - \tilde{\xi}_j^0(t)V_t\right)dt + \varepsilon\sqrt{V_t}dZ_t^{Qj+1} \end{cases}$$

Finally the model looks like the Heston model for which a well known methodology of efficient pricing based on FFT exists.

4.1.1. Moment generating function for the caplet

To price a caplet we need the moment generating function of $X_u = \log\frac{L_u^j}{L_t^j}$. We denote by

$$\phi(t, X_t, V_t, z) = E^{Qj+1}[e^{zX_{T_j}}|\mathcal{F}_t].$$

This function $\phi(t, x, V, z)$ satisfies, using Feynman-Kac argument, the partial differential equation

$$\begin{cases} \partial_t\phi + \kappa\left(\theta - \tilde{\xi}_j^0(t)V\right)\partial_V\phi - \frac{1}{2}\|\gamma_t^j\|^2 V\partial_x\phi + \\ \frac{1}{2}\varepsilon^2 V\partial_{VV}^2\phi + \varepsilon\rho_j(t)V\|\gamma_t^j\|\partial_{Vx}^2\phi + \frac{1}{2}\|\gamma_t^j\|^2 \\ V\partial_{xx}^2\phi = 0 \\ \phi(T, x, V, z) = e^{zx} \end{cases}$$

The characteristic function is $\phi_T(z) = \phi(t, 0, V_t, iz)$.

4.1.2. Moment generating function for the swaption

For the swaption pricing from (2), (11), (12) and using Itô's Lemma we deduce that

$$\begin{cases} dS_t^{s,M} = \sum_{j=s}^{M-1} \dfrac{\partial S_t^{s,M}}{\partial L_t^j} L_t^j\sqrt{V_t}\gamma_t^j \cdot \left[dW_t^Q - \sqrt{V_t}\sigma_S(t)dt\right] \\ dV_t = \kappa\left(\theta - \tilde{\xi}_S(t)V_t\right)dt + \varepsilon\sqrt{V_t}\left[dZ_t^Q + \xi_S(t)dt\right] \end{cases}$$

with

$$\begin{cases} \sigma_S(t) = \sum_{j=s}^{M-1}\alpha_j(t)\sigma_B(t, T_{j+1}) \\ \tilde{\xi}_S(t) = 1 + \frac{\varepsilon}{\kappa}\sum_{j=s}^{M-1}\alpha_j(t)\xi_j(t) \\ \alpha_j(t) = \dfrac{\tau B_t^{j+1}}{\sum_{j=s}^{M-1}\tau B_t^{j+1}} \\ \dfrac{\partial S_t^{s,M}}{\partial L_t^j} = \dfrac{\tau S_t^{s,M}}{(1 + \tau L_t^j)}\left(\dfrac{B_t^M}{B_t^s - B_t^M} + \dfrac{\sum_{k=j+1}^{M}\tau B_t^k}{\sum_{j=s+1}^{M}\tau B_t^j}\right) \end{cases}$$

Under the swap probability measure denoted Q^S the dynamic of the forward swap rate is given by

$$\begin{cases} dS_t^{s,M} = \sum_{j=s}^{M-1}\dfrac{\partial S_t^{s,M}}{\partial L_t^j}L_t^j\sqrt{V_t}\gamma_t^j \cdot dW_t^{Q^S} \\ dV_t = \kappa\left(\theta - \tilde{\xi}_S(t)V_t\right)dt + \varepsilon\sqrt{V_t}dZ_t^{Q^S} \end{cases}$$

with $dW_t^{Q^S} = dW_t^Q - \sqrt{V_t}\sigma_S(t)dt$ and $dZ_t^{Q^S} = dZ_t^Q - \sqrt{V_t}\xi_S(t)dt$ where $\left(W_t^{Q^S}; t \geq 0\right)$ resp. $\left(Z_t^{Q^S}; t \geq 0\right)$ is a d dimensional resp. 1 dimensional Brownian motion under Q^s. To ensure analyticity we freeze a part of the volatility of the forward swap rate and the drift of the volatility to get

$$\begin{cases} \dfrac{dS(t,T_s,T_M)}{S(t,T_s,T_M)} = \sum_{j=s}^{M-1}\omega_j(0)\sqrt{V_t}\gamma_t^j\cdot dW_t^{Q^S} \\ dV_t = \kappa\left(\theta - \tilde{\xi}_S^0(t)V_t\right)dt + \varepsilon\sqrt{V_t}dZ_t^{Q^S} \\ \omega_j(0) = \dfrac{\partial S_0^{s,M}}{\partial L_0^j}\dfrac{L_0^j}{S_0^{s,M}} \\ \tilde{\xi}_S^0(t) = 1 + \frac{\varepsilon}{\kappa}\sum_{j=s}^{M-1}\alpha_j(0)\xi_j^0(t) \end{cases}$$

As for the caplet we are interested in getting a closed form formula for the characteristic function of $X_u = \log\frac{S_u^{s,M}}{S_t^{s,M}}$. Define

$$\phi(t,X_t,V_t,z) = E^{Q^S}\left[e^{zX_T}\middle|\mathcal{F}_t\right].$$

The function $\phi(t,x,V,z)$ satisfies the PDE

$$\begin{cases} \partial_t\phi + \kappa\left(\theta - \tilde{\xi}_S^0(t)V\right)\partial_V\phi - \frac{1}{2}\left\|\gamma_{s,M}(t)\right\|^2 V\partial_x\phi \\ +\frac{1}{2}\varepsilon^2 V\partial_{VV}^2\phi + \varepsilon\rho^S(t)V \\ \left\|\gamma_{s,M}(t)\right\|\partial_{Vx}^2\phi + \frac{1}{2}\left\|\gamma_{s,M}(t)\right\|^2 V\partial_{xx}^2\phi = 0 \\ \phi(T,x,V,z) = e^{zx} \end{cases}$$

with $\gamma_{s,M}(t) = \sum_{j=s}^{M-1}\omega_j(0)\gamma_t^j$ and $\rho^S(t) = \dfrac{\sum_{j=s}^{M-1}\omega_j(0)\left\|\gamma_t^j\right\|\rho_j(t)}{\left\|\gamma_{s,M}(t)\right\|}$. Furthermore the authors suggest to approximate

$$\rho^S(t) \sim \sum_{j=s}^{M-1}\omega_j(0)\rho_j(t).$$

In fact, this approximation is not needed because only $\rho^S(t)\left\|\gamma_{s,M}(t)\right\|$ appears in the PDE and it is related to $\left\|\gamma_t^j\right\|\rho_j(t)dt$ already known therefore we will not follow the authors. We define the characteristic function $\phi_T(z)$ by $\phi_T(z) = \phi(t,0,V_t,iz)$.

4.1.3. Computing the moment generating function

The PDEs are identical therefore we write them in a compact form

$$\begin{cases} \partial_t\phi + \kappa(\theta - \beta(t)V)\partial_V\phi - \frac{1}{2}\lambda(t)^2 V\partial_x\phi \\ +\frac{1}{2}\varepsilon^2 V\partial_{VV}^2\phi + \varepsilon\rho(t)V\lambda(t)\partial_{Vx}^2\phi + \frac{1}{2}\lambda(t)^2 \\ V\partial_{xx}^2\phi = 0 \\ \phi(T,x,V,z) = e^{zx} \end{cases}$$

	$\beta(t)$	$\lambda(t)$	$\rho(t)$	$\zeta(t)$
caplet	$\xi_j^0(t)$	$\left\|\gamma_t^j\right\|$	$\rho_j(t)$	$\left\|\gamma_t^j\right\|\rho_j(t)$
swaption	$\tilde{\xi}_S^0(t)$	$\left\|\gamma_{s,M}(t)\right\|$	$\rho^S(t)$	$\rho^S(t)\left\|\gamma_{s,M}(t)\right\|$

where the time dependency of the parameters is emphazised. Looking for a solution of the form $\phi(t,x,V,z) = e^{A(t,z)+B(t,z)V+zx}$ we obtain the Riccati equations

$$\begin{aligned} -\partial_t A(t,z) &= \kappa\theta B(t,z) \\ -\partial_t B(t,z) &= \frac{1}{2}\varepsilon^2 B(t,z)^2 + (\rho(t)\varepsilon\lambda(t)z - \kappa\beta(t))B(t,z) \\ &\quad +\frac{1}{2}\lambda(t)^2(z^2 - z) \\ &= b_2(t)B(t,z)^2 + b_1(t)B(t,z) + b_0(t) \end{aligned}$$

with terminal conditions $A(T,z) = 0$ and $B(T,z) = 0$.

Under the hypothesis that the volatility is piecewise constant and the maturity of the option is T_N the solution of the above System is given by

$$\begin{cases} B(t,z) = B(T_{i+1},z) + \dfrac{-b_1 + d - 2B(T_{i+1},z)b_2}{2b_2\left(1 - ge^{d(T_{i+1}-t)}\right)} \\ \left(1 - e^{d(T_{i+1}-t)}\right) \\ A(t,z) = A(T_{i+1},z) + \dfrac{a_0}{2b_2} \\ \left((-b_1 + d)(T_{i+1} - t) - 2\ln\left(\dfrac{1 - ge^{d(T_{i+1}-t)}}{1-g}\right)\right) \end{cases}$$

for $t \in [T_i, T_{i+1}]$ and $i \in \{0,\ldots,N-1\}$ with

$$\begin{aligned} A(T_N,z) &= 0 \\ B(T_N,z) &= 0 \\ a_0 &= \kappa\theta \\ b_1 &= \rho(T_i)\varepsilon\lambda(T_i)z - \kappa\beta(T_i) \\ b_0 &= \frac{\lambda(T_i)^2}{2}(z^2 - z) \\ b_2 &= \frac{\varepsilon^2}{2} \\ d &= \sqrt{\Delta} \\ \Delta &= b_1^2 - 4b_0b_2 \\ g &= \frac{-b_1 + d - 2B(T_{i+1},z)b_2}{-b_1 - d - 2B(T_{i+1},z)b_2} \end{aligned}$$

Remark: For computational purpose we embed the caplet/floorlet structure in the swaption structure. In fact we have $L_t^i = S_t^{i,i+1}$ as such for pricing a caplet or a swaption we will use the same algorithm.

4.2. DERIVATIVES PRICING

For the caplet $Cplt(t,T_M,K,\tau,N)$ we have

$$\begin{aligned} Cplt(t,T_M,K,\tau,N) &= B_t^{M+1}\tau\, N\, E_t^{M+1}\left[\left(L_{T_M}^M - K\right)_+\right] \\ &= B_t^{M+1}\tau\, N\, L_t^{Q^{M+1}}\left(I_1 - \frac{K}{L_t^{M+1}}I_2\right) \end{aligned}$$

with

$$I_1 = E_t^{Q^{M+1}}\left[e^{\log\frac{L_{T_M}^M}{L_T^M}}1_{\left\{\frac{L_{T_M}^M}{L_T^M} > \frac{K}{L_T^M}\right\}}\right]$$

$$I_2 = E_t^{Q^{M+1}}\left[1_{\left\{\frac{L_{T_M}^M}{L_T^M} > \frac{K}{L_T^M}\right\}}\right]$$

For the European payer swaption $Swpt(t,T_s,T_M, K,\tau,N)$

$$Swpt(t, T_s, T_M, K, \tau, N) = \sum_{i=s}^{M-1} B_t^{i+1} \tau N S_t^{s,M} \left(I_1 - \frac{K}{S_t^{s,M}} I_2 \right)$$

$$I_1 = E_t^{QS}\left[e^{\log \frac{S_{T_s}^{s,M}}{S_t^{s,M}}} 1 \left\{ \frac{S_{Ts}^{s,M}}{S_t^{s,M}} > \frac{K}{S_t^{s,M}} \right\} \right]$$

$$I_2 = E_t^{QS}\left[1 \left\{ \frac{S_{Ts}^{s,M}}{S_t^{s,M}} > \frac{K}{S_t^{s,M}} \right\} \right]$$

The floorlet $Flt(t, T_M, K, \tau, N)$ or the European receiver swaption can be computed in similar way.

Computing the integrals

We have the following expressions for I_1 and I_2

$$I_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \frac{\mathrm{Im}\left\{ e^{-iu \log\left(\frac{K}{X(t)}\right)} \phi_T(1 + iu) \right\}}{u} du$$

$$I_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \frac{\mathrm{Im}\left\{ e^{-iu \log\left(\frac{K}{X(t)}\right)} \phi_T(iu) \right\}}{u} du$$

where $\phi_T(u)$ is conveniently choosed whether a caplet or a swaption is priced and $X(t) = L_t^M$ resp. $X(t) = S_t^{s,M}$ for the caplet/floorlet resp. the swaption (receiver or payer). It is also possible to compute the price using FFT method as in Carr and Madan [6], the computation speed is approximatively twice faster.

4.3. NUMERICAL EXAMPLES

For our numerical experiments we choose a two factors model with the following piecewise volatility structure: $\gamma_t^k = (\gamma_t^{1,k}, \gamma_t^{2,k})$.

if $t \in [T_j, T_{j+1}[$

$$\gamma_t^{1,k} = 0.2$$

$$\gamma_t^{2,k} = \frac{0.01 - 0.05e^{-0.1(j-k)}}{\sqrt{0.04 + 0.00075j}}$$

and

$$dW_t^{1,Q} dZ_t^Q = \rho^1 dt = 0.5dt$$

$$dW_t^{2,Q} dZ_t^Q = \rho^2 dt = 0.2dt$$

the yield curve is flat at 5%, $Vo = 1$, $\varepsilon = 0.6$, $\kappa = 1$ and $\theta = 1$.

Remark on the qualitative features of this model:

- produces a smile because of the stochastic volatility process
- produces a skew controled by the correlation between $(W_t^Q; t \geq 0)$ and $(Z_t^Q; t \geq 0)$.

In this section we presented an implementation of the stochastic volatility Libor Market Model as proposed by Wu and Zhang [25]. It admits a closed form solution for the characteristic function of the Libor resp. swap rate thus an efficient pricing procedure can be carried out for the Caplet/Floorlet resp. swaption. Further study is needed in order to test the calibration capabilities of this model. This important aspect will be handled in a future release of Premia (see Privault and Wei [24]).

V. Malliavin calculus

In this part we present an application of Malliavin calculus to the computation of the Greeks within the LMM. The computation of sensitivities is of tremendous importance for risk management purpose as the hedging strategy relies on them. For high dimensional products (with several underlyings) the price and the hedging ratios are computed by Monte Carlo method. For products with irregular payoff this often leads to unreliable values for the hedging ratios. For equity derivatives an approach based on the Malliavin calculus was proposed in order to reduce these problems. In this part we present a strategy to apply the Malliavin calculus to interest rate derivatives products even if the hypothesis of ellipticity (which is usually put forward in equity derivatives) is not satisfied.

Following Lions *et al.* [12] we present the theoretical aspects of the Malliavin calculus, we refer also to Nualart [21] or Bally [1]. Then we recall how the integration by part formula is used to compute the sensitivities for equity derivatives products. It will allow us to underline the difficulties when dealing with interest rates derivatives. Then

Table 2 : Swaption payer prices in bps

swaption maturity	Tenor	price(strike 0.8 ATM)	price(strike ATM)	price(strike 1.2 ATM)
1	1	114.683	64.519	34.655
1	5	609.080	405.221	267.585
1	10	1472.062	1179.612	954.980
3	1	151.380	116.830	91.083
3	5	869.485	739.835	636.496
3	10	2201.807	2057.297	1934.698
5	1	185.766	161.735	142.306
5	5	1087.164	1009.870	944.592
5	10	2257.460	1904.210	1623.445

we propose a solution to carry out the integration by part formula and illustrate how it works when Computing the delta and the gamma for an European swaption. For other examples we refer to Da Fonseca and Messaoud [11].

5.1. THE MATHEMATICAL FRAMEWORK

Let $(W_t\,;t \geq 0)$ a d-dimensional Brownian motion on a complete probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ is the filtration generated by $(W_t\,;t \geq 0)$. Let $\mathcal{C}$ the set of random variables F of the form

$$F = \phi\left(\int_0^T h_1(t)dW_t, \ldots, \int_0^T h_d(t)dW_t\right)$$

for ϕ in $\mathcal{S}(\mathbb{R}^d)$ the set of infinitely differentiable and rapidly decreasing function on $\mathbb{R}^d$ and $h_1, \ldots, h_d \in L(\Omega \times [0,T])$. For F *in* $\mathcal{C}$ the Malliavin derivative DF of F is defined as the process $(D_t F\,; t \geq 0)$ of $L(\Omega \times [0,T])$ with values in $L([0,T])$ *by*

$$D_t F = \sum_{i=1}^{d} \partial_{xi}\phi \left(\int_0^T h_1(t)dW_t, \ldots, \int_0^T h_d(t)dW_t\right) h_i(t), t \geq 0 \; a.s.$$

We define the norm on $\mathcal{C}$

$$\| F \|_{1,2} = (E(F^2))^{\frac{1}{2}} + \left(E\left(\int_0^T (D_t F)^2 dt\right)\right)^{\frac{1}{2}},$$

and denote by $\mathbf{D}_{1,2}$ the Banach space which is the completion of $\mathcal{C}$ with respect to the norm $\| \,.\, \|_{1,2}$.

Property 1. *Let* $f : \mathbb{R}^d \to \mathbb{R}$ *be a continuously differentiable function with bounded partial derivatives and* $F = (F_1, \ldots, F_d)$ *a random vector with* $F_i \in \mathbf{D}_{1,2}$. *Then* $f(F) \in \mathbf{D}_{1,2}$ *and*

$$D_t f(F) = \sum_{i=1}^{s} \partial_{xi} f(F) D_t F_i, \quad t \geq 0 \; a.s.$$

5.1.1. Integration by parts formula

Definition 5.1. *The Skorohod integral denoted by* δ *is defined on*

$$dom(\delta) = \left\{ \begin{array}{l} u \in L^2(\Omega \times [0,T]) : E\left[\int_0^T D_t F_{ut}\, dt\right] \leq \\ C(u) \| F \|_{1,2} \quad \forall F \in \mathbf{D}_{1,2} \end{array} \right\}.$$

If $u \in dom(\delta)$, *we define* $\delta(u)$ *by:*

$$E(\phi\delta(u)) = E(< Du, u >) \forall \phi \in \mathbf{D}_{1,2} \qquad (14)$$

Remark 5.1. *For adapted processes the Skorohod integral coincides with Itô integral.*

Proposition 5.1. *Suppose* $F \in \mathbf{D}_{1,2}$, $h \in L^2([0,T])$ *then*

$$E[< D_t F, h >] = E\left[F\int_0^T h_t dW_t\right]. \qquad (15)$$

Lemma 5.1. $F, G \in \mathbf{D}_{1,2}$, $h \in L^2([0,T])$ *then*

$$E[G < D_t F, h >] = E[-F < D_t G, h >] + E[FGW(h)].$$

Proposition 5.2. *Let* $F \in \mathbf{D}_{1,2}$, $h \in dom(\delta)$

$$\delta(Fh) = F\delta(h) - \int_0^T D_t F h_t\, dt. \qquad (16)$$

The Malliavin calculus has been recently used in the computation of the price sensitivities of financial derivative products, the main article being Lions *et al.* [12]. Let us recall briefly the known results.

5.1.2. Computation of sensitivities

We recall the main results on the use of Malliavin calculus for the computation of the Greeks to emphasize the problem we face when we want to apply this tool to interest rate derivatives. Essentially we rely on proposition 3.2 page 399 of Lions *et al.* [12].

We denote by $(X_t\,; 0 \leq t \leq T)$ the multidimensional vector in $\mathbb{R}^d$ representing the underlying assets and solution of the following SDE :

$$dX(t) = b(X_t)dt + \sigma(X_t)dW_t,$$

where $(W_t\,; 0 \leq t \leq T)$ is a Brownian motion in $\mathbb{R}^d$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t$ is the filtration generated by $(W_t\,; t \geq 0)$. We denote by $(Y_t\,; 0 \leq t \leq T)$ the first tangent process associated to X_t as the solution of the following SDE:

$$Y(0) = I_n$$

$$dY(t) = b'(X_t)Y(t)dt + \sum_{i=1}^{n} \sigma_i'(X_t)Y(t)dW_t^i,$$

where I_d is the identity matrix of order d, b' is the derivative matrix of b and σ_i' is the derivative matrix of the i^{th} column of σ. We can also prove the following result (see Nualart [21] or Lions *et al.* [12] Property P2 page 395) linking the Malliavin derivative to the tangent process:

$$D_s X_T = Y(T) Y_s^{-1} \sigma(X(s)).$$

Assuming that there is no interest rate, for a payoff $\Phi(X_t)$ the price of the option is given by

$$u(x) = E\left[\Phi(X_T) | X_0 = x\right] = E^x[\Phi(X_T)],$$

where E is the expectation under the risk neutral probability P.

The computation of the delta by mean of Malliavin calculus gives:

$$\frac{du(x)}{dx} = E^x\left[\Phi(X_T)\int_0^T < \sigma^{-1}(X_t)Y_t, dW_t >\right]. \qquad (17)$$

This result must be compared with pathwise approach that gives

$$\frac{du(x)}{dx} = E^x[\Phi'(X_T)Y_T]. \qquad (18)$$

with Φ' being the derivative of Φ.

The main advantage of (17) compared (18) relies in the fact that we need not to differentiate the payoff function, a fact that is crucial when dealing with non smooth payoff.

Another point is the fact we need to invert the matrix σ of the assets to get (17).

Now if we turn to the Libor market model framework, we see that we can not apply this approach. In this case we have a non invertible matrix σ since we have less factors than underlyings, σ is a non square matrix. More precisely the diffusion is not elliptic, a condition needed to apply the methodology proposed by Lions *et al.* [12] that allows to compute the Greeks without differenciating the payoff. It is an unfortunate situation as irregular payoffs are very common in the interest rate derivatives market.

5.1.3. Processes involved in the computations

In this part we collect all processes involved in the computation of the sensitivities for a European swaption within the LMM.

Using the definition of / given in (4) simple calculus lead to $\partial_{L_t^i} f = \sum_{j=s}^{M-1} \partial_j f \partial_{L_t^i} L_{T_s}^j = \partial_j f \partial_{L_t^i} L_{T_s}^j$ where the tangent process $\partial_{L_t^i} L_{T_s}^j$ is computed from the equation (3) and has the following expression

$$\partial_{L_t^i} L_{T_s}^j = \delta_{ij} \frac{L_{T_s}^j}{L_t^j} + \partial_i F_{T_s}^j L_{T_s}^j 1_{\{i \leq j\}}.$$

Furthermore knowing the dynamic of the Libor rates (3) it is straightforward to show that $D_u^m L_{T_s}^j = L_{T_s}^j \gamma_u^{j,m}$ for $u \leq T_s$ and we will denote the malliavin derivative of $L_{T_s}^j$ by $D_u L_{T_s}^j = \left(D_u^1 L_{T_s}^j, \ldots, D_u^d L_{T_s}^j\right)^*$ which can be written as $D_u L_{T_s}^j = L_{T_s}^j \gamma_u^j$. Finally the Malliavin derivative of f is $D_u f = (D_u^1 f, \ldots, D_u^d f)^*$ with $D_u^m f = \sum_{j=s}^{M-1} \partial_j f D_u^m L_{T_s}^j = \partial_j f L_{T_s}^j \gamma_u^{j,m}$. Also we will need the second Malliavin derivative of / given by

$$D_u^m D_v^n f = \partial_{jk}^2 f L_{T_s}^j L_{T_s}^k \gamma_u^{j,m} \gamma_v^{k,n} + \partial_j f L_{T_s}^j \gamma_u^{j,m} \gamma_v^{j,n}.$$

5.2. COMPUTING THE DELTA

Definition 5.2. *We define the i^{th} delta as the first derivative of the price with respect to the initial condition of Libor with maturity T_i*

$$\Delta^i := \partial_{L_t^i} E\left[\Phi\left(f\left(L_{T_s}^j\right); j = s..M-1\right)\right]$$

Proposition 5.3. *Given a stochastic process h_u not necessarily adapted we have an integration by part formula (IBP) that allows us to compute the Delta. In fact we have*

$$\Delta^i = E[\Phi(.) H^{i,h}]$$

with

$$H^{i,h} = \frac{\partial_{L_t^i} f}{\int_t^{T_s} h_v \cdot D_v f dv} \delta(h) - \int_t^{T_s} h_u \cdot D_u \left(\frac{\partial_{L_t^i} f}{\int_t^{T_s} h_v \cdot D_v f dv} \right) du.$$

Proof.

$$\begin{aligned} \Delta^i &= E\left[\Phi'(.) \partial_{L_t^i} f\right] = \\ & E\left[\Phi'(.) \frac{\int_t^{T_s} h_u \cdot D_u f du}{\int_t^{T_s} h_v \cdot D_v f dv} \partial_{L_t^i} f\right] \\ &= E\left[\int_t^{T_s} \frac{D_u \Phi(.) \cdot h_u}{\int_t^{T_s} h_v \cdot D_v f dv} \partial_{L_t^i} f du\right] = \\ & E\left[\Phi(.) \delta \frac{\partial_{L_t^i} f}{\int_t^{T_s} h_v D_v f dv} h.\right] \\ &= E\left[\Phi(.) \frac{\partial_{L_t^i} f}{\int_t^{T_s} h_v D_v f dv} \delta(h.)\right] - \\ & E\left[\Phi(.) \int_t^{T_s} h_u \cdot D_u \left(\frac{\partial_{L_t^i} f}{\int_t^{T_s} h_v \cdot D_v f dv}\right) du\right] \end{aligned}$$

where the different processes were defined in the preceding section. ■

At least two cases are possible for the function h_u. We denote by $H^{i,1}$ the weight when we choose $h_u = 1 = (1, \ldots, 1)^*$ and $H^{i,mv}$ when $h_u = D_u f$.

The Uniform estimator. In the first case we get

$$H^{i,1} = \frac{\partial_{L_t^i} f}{\int_t^{T_s} 1 \cdot D_u f dv} 1 \cdot \left(W_{T_s} - W_t\right) - \int_t^{T_s} 1 \cdot D_u \left(\frac{\partial_{L_t^i} f}{\int_t^{T_s} 1 \cdot D_v f dv} \right) du.$$

We can see that we need some positivity condition to ensure that $H^{i,1}$ is well behaved. In fact $\int_t^{T_s} 1 \cdot D_v f dv = \sum_{m=1}^d \sum_{j=s}^{M-1} \partial_j f L_{T_s}^j \int_t^{T_s} \gamma_u^{j,m} du$ and under the positivity hypothesis of the volatility, ie $\gamma_u^{j,m} > 0 \, \forall m \in \{1, \ldots, d\} j \in \{s, \ldots, M-1\} \, \forall u \in [t, T_s]$, it is easy to check the integrability of $H^{i,1}$ (recall that $\partial_j f$ is positive).

Malliavin estimator. If we take $h_u = D_u f$ we get the Malliavin weight denoted $H^{i,mv}$ whose expression is given by

$$H^{i,mv} = \frac{\partial_{L_t^i} f}{\int_t^{T_s} h_v \cdot D_v f dv} \delta(D.f) - \int_t^{T_s} D_u f \cdot D_u \left(\frac{\partial_{L_t^i} f}{\int_t^{T_s} h_v \cdot D_v f dv} \right) du.$$

Once again to transform the Skorohod integral in the right hand side into an Itô integral we use the relation (16) and we obtain

$$\begin{aligned} \delta(D.f) &= \delta\left(\partial_j f D.L_{T_s}^j\right) = \delta\left(\partial_j f L_{T_s}^j \gamma_.^j\right) \\ &= \partial_j f L_{T_s}^j \int_t^{T_s} \gamma_u^j \cdot dW_u - \\ & \int_t^{T_s} \gamma_u^j \cdot D_u\left(\partial_j f L_{T_s}^j\right) du. \end{aligned}$$

Finally the expression for $H^{i,mv}$ is

$$H^{i,mv} = \frac{\partial_{L_t^i} f}{\int_t^{T_s} D_v f \cdot D_v f dv} = \Big(\partial_j f L_{T_s}^j \int_t^{T_s} \gamma_u^j \cdot dW_u - \int_t^{T_s} \gamma_u^j \cdot D_u \big(\partial_j f L_{T_s}^j\big) du\Big) - \int_t^{T_s} D_u f \cdot D_u \left(\frac{\partial_{L_t^i} f}{\int_t^{T_s} D_v f \cdot D_v f dv}\right) du. \quad (19)$$

This estimator allows a larger class of volatility function. In fact, we need some positivity of the term $\int_t^{T_s} D_v f \cdot D_v f dv = \int_t^{T_s} \sum_{m=1}^{d} \Big(\sum_{j=s}^{M-1} \partial_j f L_{T_s}^j \gamma_u^{j,m}\Big)^2 du$. For a reasonable volatility structure not necessarily with all positive components (for example a positive first factor for the volatility) it is easy to check the integrability of $H^{i,mv}$ (recall that $\partial_j f$ is positive and does not depend on the volatility structure).

Denote by I the equation (19), it can be computed explicitly to get

$$I = \frac{1}{\int_t^{T_s} D_v f \cdot D_v f dv} \int_t^{T_s} D_u f \cdot D_u \big(\partial_{L_t^i} f\big) du - \frac{\partial_{L_t^i} f}{\Big(\int_t^{T_s} D_v f \cdot D_v f dv\Big)^2} \int_t^{T_s} D_u f \cdot D_u \Big(\int_t^{T_s} D_v f \cdot D_v f dv\Big) du,$$

with

$$\begin{aligned}
D_u\big(\partial_j f L_{T_s}^j\big) &= \partial_{jk}^2 f L_{T_s}^j L_{T_s}^k \gamma_u^k + \partial_j f L_{T_s}^j \gamma_u^j \\
D_u\big(\partial_{L_t^i} f\big) &= \partial_{jk}^2 f \partial_{L_t^i} L_{T_s}^j L_{T_s}^k \gamma_u^k + \partial_j f \partial_{L_t^i} L_{T_s}^j \gamma_u^j \\
\int_t^{T_s} D_v f \cdot D_v f dv &= \sum_{k,j=s}^{M-1} \partial_j f L_{T_s}^j \partial_k f L_{T_s}^k \int_t^{T_s} \gamma_u^j \cdot \gamma_u^k du \\
&= \partial_j f L_{T_s}^j \partial_k f L_{T_s}^k (\gamma^j, \gamma^k)
\end{aligned}$$

$$\begin{aligned}
\int_t^{T_s} \gamma_u^i \cdot D_u\big(\partial_j f L_{T_s}^j\big) du &= \sum_{m=1}^{d} \sum_{j=s}^{M-1} \int_t^{T_s} \gamma_u^{j,m} D_u^m \big(\partial_j f L_{T_s}^j\big) du \\
&= \partial_{jk}^2 f L_{T_s}^j L_{T_s}^k (\gamma^j, \gamma^k) + \partial_j f L_{T_s}^j (\gamma^j, \gamma^j) \\
\int_t^{T_s} D_u f \cdot D_u\big(\partial_{L_t^i} f\big) du &= \partial_j f L_{T_s}^j \partial_{lk}^2 f L_{T_s}^k \partial_{L_t^i} L_{T_s}^l (\gamma^j, \gamma^k) + \partial_j f L_{T_s}^j \partial_k f \partial_{L_t^i} L_{T_s}^k (\gamma^j, \gamma^k)
\end{aligned}$$

$$\begin{aligned}
\int_t^{T_s} D_u f \cdot D_u \Big(\int_t^{T_s} d_v f \cdot D_v f dv\Big) &= 2 \partial_j f L_{T_s}^j \partial_{\ln}^2 f L_{T_s}^n \partial_k f L_{T_s}^l L_{T_s}^k (\gamma^l, \gamma^k)(\gamma^j, \gamma^n) \\
&\quad + 2 \partial_j f L_{T_s}^j \partial_l f L_{T_s}^l \partial_k f L_{T_s}^k (\gamma^k, \gamma^j)(\gamma^l, \gamma^k).
\end{aligned}$$

5.3. NUMERICAL RESULTS

The experiments on the delta show that the Malliavin estimator gives the same accuracy as the finite difference one, see figures (1) and (2)[7]. This is always the case on European call option. We remark also that using many Libor rates in a swaption play a smoothing rule in the Malliavin estimator. Actually the variance of this estimator is higher when using less Libor rates.

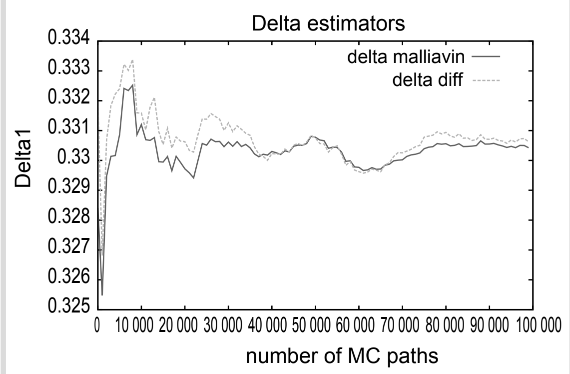
5.4. COMPUTING THE GAMMA

Definition 5.3. *We define the gamma of the swaption as the second derivative with respect to the intial forward rate curve. It is given by*

$$\Gamma_{ij} = \partial_{L_t^i L_t^j}^2 E_t[\Phi(f(.))]$$

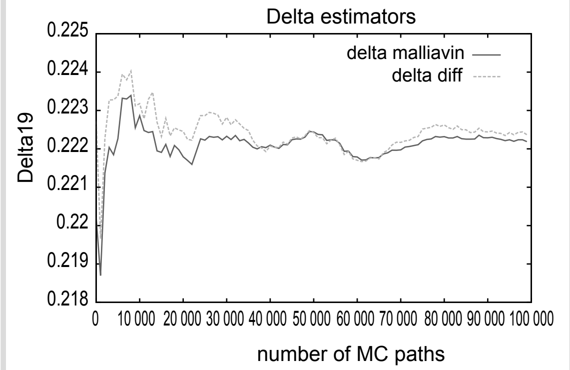
Another possible choice is given by $\partial_{B_t^i B_t^j}^2 E_t[\Phi(f(.))]$ but it is easy to relate those quantifies. For practical purpose the gamma is important as it gives information on product convexity and delta hedging errors. We usually

Figure 1. Delta defined by $\partial_{L_t^{s+2}} E[\Phi(f(.))]$



Variance of the estimators for a number of paths ranging from 1000 to 100000.

Figure 2. Delta defined by $\partial_{L_t^{s+20}} E[\Phi(f(.))]$



Variance of the estimators for a number of paths ranging from 1000 to 100000.

evaluate the gamma by finite difference method, a central scheme leads to

$$\partial^2_{L_t^i L_t^i} E_t[\Phi(f(.))] \sim \frac{E_t\big[\Phi\big(f(L_t^i+\varepsilon)\big)\big] - 2E_t\big[\Phi\big(f(L_t^i)\big) + E_t\big[\Phi\big(f(L_t^i-\varepsilon)\big)\big]\big]}{\varepsilon^2}$$

where $E_t\big[\Phi\big(f(L_t^i+\varepsilon)\big)\big]$ stands for the expectation evaluated with inital value for the i^{th} forward rate equals to $L_t^i+\varepsilon$. It is well known that accuracy of the method relies on the choice of ε which is very difficult. For the cross gamma defined by $\partial^2_{L_t^i L_t^j} E_t[\Phi(f(.))]$ a finite difference approximation estimator is also available.

It is well known that for European vanilla equity derivative the finite difference gamma estimator has a large variance Lions *et al.* [12]. It is mainly due to the fact that the second derivative of the payoff $(x \to (x-K)_+)$ is the Dirac function. Using Malliavin calculus we can integrate the payoff and derive the density of the underlying random variable, thus leading to a function less irregular.

Using the results for the delta we obtain the following estimator for the gamma.

Proposition 5.4. *Given a stochastic process h_u not necessarily adapted we have an integration by part formula (IBP) that allows us to compute the Gamma. In fact we have*

$$\Gamma^{ij} = E_t[\Phi(.)G^{ij,h}]$$

with

$$G^{ij,h} = \frac{\partial_{L_t^j} fH^{i,h}}{\int_t^{T_s} h_v \cdot D_v f dv}\delta(h.) - \int_t^{T_s} h_u \cdot D_u\left(\frac{\partial_{L_t^j} fH^{i,h}}{\int_t^{T_s} h_v \cdot D_v f dv}\right)du + \partial_{L_t^j} H^{i,h}.$$

Proof.

$$\Gamma^{ij} = \partial_{L_t^j} E[\Phi(.)H^{i,h}] = E\big[\Phi'(.)\,\partial_{L_t^j} fH^{i,h}\big] + E\big[\Phi(.)\,\partial_{L_t^j} H^{i,h}\big].$$

The first expectation can be written as

$$E\big[\Phi'(.)\,\partial_{L_t^j} fH^{i,h}\big] = E\left[\Phi()\left(\frac{\partial_{L_t^j} fH^{i,h}}{\int_t^{T_s} h_v \cdot D_v f dv}\delta(h.) - \int_t^{T_s} h_u \cdot D_u\left(\frac{\partial_{L_t^j} fH^{i,h}}{\int_t^{T_s} h_v \cdot D_v f dv}\right)du\right)\right].$$

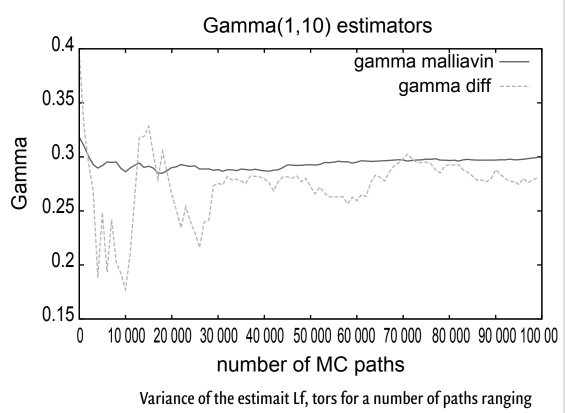
As for the delta we define two estimators. We denote by $G^{ij,1}$ the weight when we choose $h_u = 1 = (1,\ldots,1)^*$ and $G^{ij,mv}$ when $h_u = D_u f$. The Skorohod integral $\delta(h.)$ involved in the weight can be transformed into an Ito integral using the formula (16). Furthermore the Malliavin derivative of $H^{i,mv}$ implies the computation of $D^{m1}_{u_1} D^{m2}_{u_2} D^{m3}_{u_3} f$ which is straightforward.

5.5. NUMERICAL RESULTS

We compute the gamma for a 10 years in 5 years European swaption for 6 months Libor rates using the Malliavin estimator and compare it with a finite difference method estimator. In figure (3) we report the values for the cross gamma for different number of Monte Carlo paths[8]. It shows that the Malliavin estimator outperforms the finite difference estimator which presents a large variance. This result is consistent with those obtained in equity derivatives and is mainly due to the fact that the computation of the gamma involves an expectation of a very irregular function (the Dirac function). The same conclusions can be done for the gamma with respect to one Libor rate as it can be seen in figure (4). The difference between the two estimators is even more important.

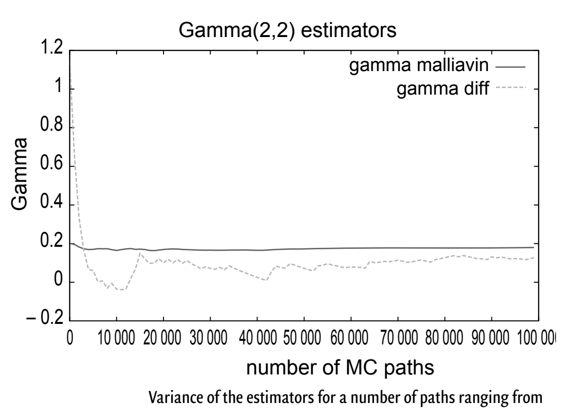
Remarks: the advantage of the Malliavin calculus (the integration by part formula) can appear for first order sensitivity if the payoff is sufficiently irregular. For example the TARN (target redemption accrual note) is a product whose payoff looks like a digital. In that case the

Figure 3. Cross gamma defined by $\partial^2_{L_t^S L_t^S + 10} E[\Phi(f(.))]$



Variance of the estimait Lf, tors for a number of paths ranging from 1000 to 100000.

Figure 4. Gamma defined by $\partial^2_{L_t^S + 2} E[\Phi(f(.))]$



Variance of the estimators for a number of paths ranging from 1000 to 100000.

estimator of the delta, the sensitivity with respect to the initial Libor curve, computed using Malliavin calculus is much more efficient (less variance) than the one obtained using finite difference method. The advantage carries out also to the vega the sensitivity with respect to the volatility factors. We refer to Da Fonseca and Messaoud [11] for further details.

VI. Conclusion

In this paper we presented some parts of Premia 7 devoted to the Libor Market Model. The first one deals with the pricing of Bermudan swaption by means of least squares method. The application of this method to interest rate derivatives is not straightforward, the choice of a convenient numeraire is of importance for the well behavior of the methodology. Furthermore selecting the good state variables on which the regression is done is also a key issue. The choice of the numeraire and the state variables is more an art than a science and strongly depend on the product at hand.

On a second part we presented a stochastic volatility extension of the LMM. This model produces a smile for both the caplet/floorlet and swaption market and relies on the FFT. Further study is needed in order to verify if this model can handle the different forms of smile observed on the market. A future release of Premia will address the calibration issue and maybe clarify the capabilities of this model.

On a third and last part we presented some results on the application of the Malliavin calculus to the Libor market model. This technique is very useful when computing the Greeks (the sensitivities of the product). We show that even if the ellipticity condition is not satisfied by the LMM we can still apply an integration by part formula. This is particularly helpful when the payoff presents some irregularity, a situation very common in this market. We illustrate the robustness of our estimators using a vanilla swaption but the proposed methodology works also for more exotic products (see Da Fonseca and Messaoud [11]).

For a more in depth presentation of these results and for other applications, not included in this document for conciseness reasons, as well as the description of the application programming interface of Premia 7 we refer to Barton-Smith *et al.* [2]. ■

1. The transposition is denoted by *
2. The scalar product between two vectors of $\mathbb{R}^d$ x and y is written $x \cdot y = \Sigma_{i=1}^{d} x_i y_i$.
3. A cap resp. floor is a sum of caplets resp. floorlets.
4. See for instance Pedersen [22] for details.
5. For simplicity of notation, we omit functional dependence on K, τ and $\overline{N}$, considered as fixed throughout the following.
6. It is sufficient to set $T_{\bar{i}} = T_{e-1}$.
7. We use the same Brownian sample path for both estimators.
8. We use the same Brownian sample path for both estimators.

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