

RISK MANAGEMENT OF CPPI FUNDS IN SWITCHING REGIME MARKETS



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I. INTRODUCTION

The constant proportion portfolio insurance is a dynamic strategy of investment protecting a portfolio of stocks against the downside risk but keeping an upside potential. The origin of this management technique can be found in the seminal works of Merton (1971), who optimized the investment-consumption strategy maximizing the economic utility of consumption and terminal wealth. To summarize, the CPPI method consists to maintain the risk exposure equal to a constant multiple of the excess of wealth over a floor. The CPPI was introduced by Perold and Sharpe (1988) for a portfolio of bonds and revisited by Black and Jones (1987) for equities. In Black and Perold (1992), it is shown that in a complete market, the CPPI can be characterized as expected utility maximization when the utility function is HARA. The properties of continuous-time CPPI strategies are studied extensively in the literature. A comparison of OBPI and CPPI is provided in Bertrand and Prigent (2005). The literature also deals with the effects of jump processes, stochastic volatility models and extreme value approaches on the CPPI method, cf. Bertrand and Prigent (2002, 2003).

The contribution of our paper is to analyze the influence of switches of assets regimes on the CPPI performance and risk exposure. Modeling the stocks by a switching regime process is motivated by the observation that stocks returns often evolve from time to time between a stable low-volatility state and a more unstable high volatility regime. In the studied framework, the market value of the risky asset is ruled by a geometric Brownian dynamics, whose the mean and standard deviation may jump from one set of values to another one. Similar switching regime models have first been investigated by Hamilton (1989) to model the discrete shifts that can occur in the growth rate of non stationary series. This kind of models has been successfully applied to model stock returns (see e.g. Hardy (2001)) and to price options (Naik (1993)).

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Under the fairly standard assumptions of frictionless markets, we review the features of a fund managed with the CPPI strategy, in a continuous time framework. Under the additional assumption that the dynamics of the risky asset is driven by a hidden Markov process, we next propose analytical formulas to calculate the first fourth centered moments of the market value of a CPPI fund. By adopting the methodology of Carr and Madan (1998), we develop a Fast Fourier Transform approach (noted FFT in the sequel) to compute the density of the market value of a CPPI fund. Based upon this result, we show how to calculate the two main measures of short-fall risk, namely the Value At Risk (VaR) and the TailVaR. As numerical application, a 3 states switching regime model has been fitted to daily return of the CAC 40 (the French main stocks index) and expected returns, VaR, TailVaR have been computed for a one year time horizon. Our analysis reveals that if the CPPI multiple is not sufficiently high, the CPPI average return is lower than the risk free rate.

The outline of this paper is as follows. Section 2 briefly reviews the CPPI strategy. The sections 3 and 4 present the characteristics of the dynamics of the switching regime process ruling the risky assets and the calibration procedure. In section 6, the dynamics of the cushion and the moments of a CPPI fund are established. The paragraph 7 introduces the FFT method required to compute the VaR and TailVaR. The paper is ended by a numerical application to the French market.

II. CPPI

The CPPI strategy is usually used by a fund manager to protect a fund tracking a benchmark against the short-fall risk. In our model, the fund manager invests in two categories of assets: cash and a portfolio of risky assets, tracking an index. The market value of the cash account and of the portfolio of risky assets are respectively noted B_t and S_t . The dynamics of those assets is developed in the next section. The total market value of the fund, at time t , is noted V_t . The CPPI approach consists to

control continuously the exposure in risky assets so as to prevent the fund against a fall below a predetermined floor. The floor at time t will be noted $F(t)$ in the remainder of this paper. The quantity of risky assets purchased at time t , π_t , is a multiple of the difference between the market value of the fund V_t and the floor $F(t)$. This difference is called in the literature, the cushion and we denote it C_t :

$$C_t = V_t - F(t) \quad (1)$$

The quantity of stocks π_t held in portfolio is equal to m , the multiplier which is a parameter chosen by the fund manager, times the cushion:

$$\pi_t = m.C_t \quad (2)$$

Eq. (2) directly reveals that if the fund is continuously rebalanced between cash and stocks, the market value will never fall below the floor. We will first detail the dynamics of assets and next determine the expected return and volatility of such kind of fund. The multiplier m may be interpreted as the speed at which we sell and purchase stocks. The higher is the multiplier, the larger is the amount of stocks purchased when the market prices rise.

III. SWITCHING REGIMES

We assume that the portfolio of risky assets S_t , tracking an index, is ruled by a geometric Brownian motion whose the mean and standard deviation depend upon a certain state of the world. The variable indicating in which state we are at time t , is a Markovian process, noted α_t , not directly observable (this kind of process is called hidden Markov process for this reason). Under the assumption that there exist N states of the world, α_t takes its value in the set $N = \{1, 2, \dots, N\}$. The filtration generated by α_t is noted G_t . The main motivation to work within this framework is to model the fact that incomes of financial securities may switch from time to time, e.g. between a stable low-volatility state and a more unstable high volatility regime. The generator of α_t is a $N \times N$ matrix Q , whose elements, noted $q_{i,j}$, satisfy the following conditions:

$$q_{i,j} \geq 0 \quad \forall i \neq j \quad \sum_{j=1}^N q_{i,j} = 0 \quad \forall i \in N. \quad (3)$$

The probabilities of transition between states between times t and $s \geq t$ are computed as the (matrix) exponential of Q :

$$P(t,s) = \exp(Q(s-t)). \quad (4)$$

The elements of the matrix $P(t,s)$ are noted $p_{i,j}(t,s)$, $i, j \in N$. And $p_{i,j}(t,s)$ is the probability of jumping from state i at time t to state j at time s :

$$p_{i,j}(t,s) = P(\alpha_s = j | \alpha_t = i) \quad i, j \in N. \quad (5)$$

The probability of being in state i at time t , noted $p_i(t)$ depends upon the initial probabilities $p_{k=1..N}(0)$ at time $t=0$, as follows:

$$p_i(t) = P(\alpha_t = i) = \sum_{k=1}^N p_k(0) p_{k,i}(0,t) \quad \forall i \in N. \quad (6)$$

However, if the Markov process has been running for a sufficiently long enough period of time, we can show that this probability is independent from the initial state:

$$p_i = \lim_{t \rightarrow +\infty} p_i(t) \quad \forall i \in N. \quad (7)$$

This is a standard property of Markov process (called the stationary property). The probabilities $p_{i=1..N}$ can be numerically computed. However, if the Markov process counts only two states, the matrix Q is of the form

$$Q = \begin{pmatrix} -q_{1,1} & q_{1,1} \\ q_{2,2} & -q_{2,2} \end{pmatrix},$$

and the stationary probabilities have a closed form expression:

$$p_1 = \frac{q_{1,1}}{q_{1,1} + q_{2,2}} \quad p_2 = \frac{q_{2,2}}{q_{1,1} + q_{2,2}} \quad (8)$$

As mentioned at the beginning of this paragraph, the market value of the replicating portfolio of risky assets, is driven by a geometric Brownian motion. If we consider a Brownian motion W_t defined on a filtration H_t , the dynamic of S_t is ruled by the following SDE:

$$\frac{dS_t}{S_t} = \mu(\alpha_t)dt + \sigma(\alpha_t)dW_t, \quad (9)$$

where $\mu(i), \sigma(i)$ are respectively the expected growth rate and the volatility of stocks in the state of the world $i \in N$. It suffices to use the Itô's lemma to show that the solution of eq. (9) is given by the following expression:

$$S_t = S_s \exp \left(\sum_{i=1}^N \int_s^t \left(\mu(i) - \frac{\sigma(i)^2}{2} \right) \delta(i, \alpha_u) du + \sum_{i=1}^N \int_s^t \sigma(i) \delta(i, \alpha_u) dW_u \right) \quad (10)$$

where $\delta(i, \alpha_t)$ is an indicator variable that is worth 1 if we are in the state of the world i , 0 else.

The return of the cash account B_t is assumed to be constant and equal to the constant risk free rate, noted r :

$$dB(t) = r B(t) dt \quad (11)$$

If $B(0) = 1$, the solution of this last equation is $B(t) = e^{rt}$. The minimum level under which the market value of the fund may not fall is also assumed to grow at risk free rate:

$$dF(t) = rF(t)dt \quad (12)$$

If the initial value of the floor is noted F_0 , the floor at time t is worth $F(t) = F_0 e^{rt}$.

We end up this paragraph by a remark about the filtrations. In the sequel, we note F_t , the filtration on which is defined the process S_t . We draw the attention of the reader that F_t is not the smallest filtration including both G_t and H_t : $F_t \neq G_t \vee H_t$. Because the process α_t is not directly observable. As at time t the state of the world is unknown, we have the relation $F_t \subset G_t \vee H_t$. This relation will be particularly useful for the developments presented in next sections.

IV. CALIBRATION OF THE SWITCHING REGIME MODEL

This section provides the necessary tools to deduce from the quotes of a stock, the parameters of the switching regime ruling S_t . The common calibration method for regime switching models is the Hamilton filter (1989), which estimates parameters by maximum likelihood estimation (MLE). The MLE is a method to estimate a set of parameters of a statistical model given empirical observations of stock returns. Let O_1, O_2, \dots, O_n be the n observed (continuous) returns of a stock, on a period of time Δt :

$$O_i = \ln \left(\frac{S_{t_{i-1} + \Delta t}}{S_{t_{i-1}}} \right) \quad i = 1, \dots, n.$$

We assume that changes of state of the world, α_t , only happen at discrete times $t_{i=1, \dots, n}$. Hence, if we are in the j^{th} state of the world at time t_{i-1} , according to eq. (9), the return on the period of time $[t_{i-1}, t_i]$ is a normal random variable $O_i : N(\mu_j \Delta t, \sigma_j \sqrt{\Delta t})$. We note Θ the set of parameters of our model

$$\Theta = \{\mu_{j=1 \dots N}, \sigma_{j=1 \dots N}, q_{j=1 \dots N, j=1 \dots N}\}$$

The log-likelihood function is defined as follows:

$$\log L = \log f(O_1 | \Theta) + \log f(O_2 | \Theta, O_1) + \log f(O_3 | \Theta, O_1, O_2) + \dots + \log f(O_n | \Theta, O_1, \dots, O_{n-1})$$

where $f(O_k | \Theta, O_1, \dots, O_{k-1})$ is the density function of the return on the k^{th} period, conditionally to a set of given model parameters Θ and to previous observations O_1, \dots, O_{k-1} . The parameters calibrating the switching regime model are the one maximizing the log-likelihood function. Hamilton has shown that the conditional density, involved in the calculation of $f(O_k | \Theta, O_1, \dots, O_{k-1})$ may be recursively calculated:

$$f(O_k | \Theta, O_1, \dots, O_{k-1}) = \sum_{i=1}^N \sum_{j=1}^N p_{i,j}(t_{k-1}, t_k | \Theta, O_1, \dots, O_{k-1}) p_{i,j}(t_{k-1}, t_k | \Theta) f(O_k | \Theta, \alpha_{t_k} = j)$$

where

$f(O_k | \Theta, \alpha_{t_k} = j)$ is the Gaussian density of the return in state j , $N(\mu_j \Delta t, \sigma_j \sqrt{\Delta t})$,

$p_{i,j}(t_{k-1}, t_k | \Theta)$ is the probability of transition, as defined by eq. (5), from state i at time t_{k-1} to state j at time t_k for the set of parameters Θ ,

$p_i(t_{k-1} | \Theta, O_1, \dots, O_{k-1})$ is the probability of being in state i at time t_{k-1} , conditionally to previous observations.

The probability $p_i(t_{k-1} | \Theta, O_1, \dots, O_{k-1})$ may be inferred recursively from $f(O_{k-1} | \Theta, O_1, \dots, O_{k-2})$ as follows:

$$p_i(t_{k-1} | \Theta, O_1, \dots, O_{k-1}) =$$

$$\frac{\sum_{j=1}^N p_j(t_{k-2} | \Theta, O_1, \dots, O_{k-2}) p_{j,i}(t_{k-2}, t_{k-1} | \Theta) f(O_{k-1} | \Theta, \alpha_{t_{k-1}} = j)}{f(O_{k-1} | \Theta, O_1, \dots, O_{k-2})} \quad (13)$$

In order to initiate the recursion, we need to determine $f(O_1 | \Theta)$. Hamilton assumes that the Markov chain has been running for a sufficiently long enough period of time, so as to apply the stationary property of Markov chains, mentioned in section 3. In particular, we get that:

$$f(O_1 | \Theta) = \sum_{i=1}^N p_i(\Theta) f(O_1 | \Theta, \alpha_{t_1} = i),$$

where $p_i(\Theta)$ are the stationary probabilities of the Markov process α_t , as defined by eq. (7), for the set of parameters Θ . An example of calibration of a two states switching regime is presented in section 7.

V. DYNAMICS OF THE CUSHION AND MOMENTS

Remember that we have defined the cushion C_t as the difference between the market value of the fund V_t and the floor $F(t)$. The dynamics of the cushion is hence given by the relation:

$$dC_t = dV_t - dF(t). \quad (14)$$

And the dynamics of the fund managed by the CPPI method is:

$$dV_t = (V_t - \pi_t) \frac{dB_t}{B_t} + \pi_t \frac{dS_t}{S_t}, \quad (15)$$

where π_t is a multiple m of the cushion C_t . From eq. (11), (15) and (12) we infer that the cushion is ruled by a geometric Brownian motion:

$$dC_t = C_t (r + m(\mu(\alpha_t) - r)) dt + m C_t \sigma(\alpha_t) dW_t. \quad (16)$$

Applying the Itô's lemma to eq. (16) leads to

$$C_t = C_0 \exp \left(\sum_{i=1}^N \int_0^t (r - mr) + m \left(\mu(i) - \frac{\sigma(i)^2}{2} \right) \delta(i, \alpha_u) du + \sum_{i=1}^N \int_0^t m \sigma(i) \delta(i, \alpha_u) dW_u \right) \quad (17)$$

Comparing eq. (17) and (10) allows us to rewrite the cushion as a function of the market value of the risky asset, S_t :

$$C_t = C_0 e^{(r-mr)t} \left(\frac{S_t}{S_0} \right)^m \quad (18)$$

And as the CPPI fund is the sum of the cushion and of the floor, we get that:

$$V_t = F(t) + C_0 e^{(r-mr)t} \left(\frac{S_t}{S_0} \right)^m \quad (19)$$

The multiple m is a parameter chosen by the fund manager. If the rebalancing is done continuously, whatever the value of m , the market value of the fund will remain above the floor F_t . The parameter m drives in fact the speed at which the fund goes away from or comes closer to the floor. This point will be illustrated in section 7 devoted to numerical applications. The parameter plays also an important role on the expected and standard deviation of the future value of CPPI fund.

To calculate moments of V_t , we need the following result:

Proposition 1. For all $\gamma \in R$, we define the matrix B_γ as follows:

$$B_\gamma = Q + \text{diag} \begin{pmatrix} \gamma \left(\mu(1) - \frac{\sigma(1)^2}{2} \right) + \frac{1}{2} \gamma^2 \sigma(1)^2 \\ M \\ \gamma \left(\mu(N) - \frac{\sigma(N)^2}{2} \right) + \frac{1}{2} \gamma^2 \sigma(N)^2 \end{pmatrix},$$

we have that

$$E \left(\left(\frac{S_t}{S_0} \right)^\gamma | F_0 \right) = E \left(\langle \exp(B_\gamma t) \delta(0); 1 \rangle | F_0 \right) = \sum_{i=1}^N p_i(0) \langle \exp(B_\gamma t) e_i; 1 \rangle \quad (20)$$

where $\delta(t) = (\delta(i, \alpha(t)) | i \in N)$ is a vector taking its values in the set of unit vectors $\{e_1, e_2, \dots, e_N\}$ and 1 is a vector of N ones.

Proof. According to equation eq. (10), we know that

$$E \left(\left(\frac{S_t}{S_0} \right)^\gamma | F_0 \right) = E \left(\exp \left(\sum_{i=1}^N \int_0^t \gamma \left(\mu(i) - \frac{\sigma(i)^2}{2} \right) \delta(i, \alpha_u) du + \sum_{i=1}^N \int_0^t \gamma \sigma(i) \delta(i, \alpha_u) dW_u \right) | F_0 \right)$$

Given that the filtration $F_t \subset G_t \vee H_t \subset G_\infty \vee H_t$, the last expectation may be rewritten as follows:

$$E \left(\left(\frac{S_t}{S_0} \right)^\gamma | F_0 \right) = E \left(E \left(\exp \left(\sum_{i=1}^N \int_0^t \gamma \left(\mu(i) - \frac{\sigma(i)^2}{2} \right) \delta(i, \alpha_u) du + \sum_{i=1}^N \int_0^t \gamma \sigma(i) \delta(i, \alpha_u) dW_u \right) | G_\infty \vee H_0 \right) | F_0 \right)$$

Conditionally to $G_\infty \vee H_0$ the term into this last expectation is lognormal, we infer that:

$$E \left(\left(\frac{S_t}{S_0} \right)^\gamma | F_0 \right) = E \left(\exp \left(\sum_{i=1}^N \int_0^t \left(\gamma \left(\mu(i) - \frac{\sigma(i)^2}{2} \right) + \frac{1}{2} \gamma^2 \sigma(i)^2 \right) \delta(i, \alpha_u) du \right) | F_0 \right) \quad (21)$$

If we denote $T_i(t, u)$, the total time spent into state i , during the interval of time $[t, u]$, the expectation eq. (21) may then be rewritten as follows:

$$E \left(\left(\frac{S_t}{S_0} \right)^\gamma | F_0 \right) = E \left(\exp \left(\sum_{i=1}^N \left(\gamma \left(\mu(i) - \frac{\sigma(i)^2}{2} \right) + \frac{1}{2} \gamma^2 \sigma(i)^2 \right) T_i(0, t) \right) | F_0 \right) \quad (22)$$

As the sum of $T_i(t, u) \forall i \in N$ is equal to the length of the interval of time, one has that $T_N(t, u) = (u - t) - \sum_{i=1}^{N-1} T_i(t, u)$. This allows us to rewrite eq. (22) as follows

$$E \left(\left(\frac{S_t}{S_0} \right)^\gamma | F_0 \right) = E \left(\exp \left(\sum_{i=1}^{N-1} \omega_i T_i(0, t) \right) | F_0 \right) \times \exp \left(\left(\gamma \left(\mu(N) - \frac{\sigma(N)^2}{2} \right) + \frac{1}{2} \gamma^2 \sigma(N)^2 \right) t \right) \quad (23)$$

where

$$\omega_i = \left(\gamma \left(\mu(i) - \frac{\sigma(i)^2}{2} \right) + \frac{1}{2} \gamma^2 \sigma(i)^2 \right) - \left(\gamma \left(\mu(N) - \frac{\sigma(N)^2}{2} \right) + \frac{1}{2} \gamma^2 \sigma(N)^2 \right) \quad i = 1 \dots N$$

The expectation (23) is calculated by the result of Bufington and Elliott (2002) about the characteristic function of the random variable $\sum_{i=1}^{N-1} \omega_i T_i(t, u)$:

$$E \left(e^{i \sum_{i=1}^{N-1} \omega_i T_i(0, t)} | G_0 \vee H_0 \right) = \langle \exp \left((Q' + i \text{diag}(\omega_1, \omega_2, \dots, \omega_{N-1}, 0)) t \right) \delta(0); 1 \rangle. \quad (24)$$

where $\delta(t)$ is the vector of $\delta(i, \alpha_t)$, $i = 1 \dots N$ and 1 is a vector of N ones. Taking the expectation of this last equation leads to the result, eq. (20).

Corollary 2. The expectation, variance, skewness of the CPPI fund are provided by the following equations

$$E(V_t | F_0) = F(t) + E(C_t | F_0) \quad (25)$$

$$V(V_t|F_0) = E(C_t^2|F_0) - E(C_t|F_0)^2 \quad (26)$$

$$S(V_t|F_0) = \frac{E(C_t^3|F_0) - 3E(C_t|F_0)E(C_t^2|F_0) + 4E(C_t^3|F_0)}{V(V_t|F_0)^{3/2}} \quad (27)$$

$$K(V_t|F_0) = \frac{E(C_t^4|F_0) - 4E(C_t^3|F_0)E(C_t|F_0) + 6E(C_t^2|F_0)E(C_t|F_0)^2 - 3E(C_t|F_0)^3}{V(V_t|F_0)^{3/2}} \quad (28)$$

where

$$E(C_t^j|F_0) = C_0^j e^{j(r-mr)t} \sum_{i=1}^N p_i(0) \langle \exp(B_{j,m}t) e_i; 1 \rangle \quad j=1,2,3,4 \quad (30)$$

This corollary is a direct consequence of proposition (1). We will see in the numerical application that the variance is directly proportional to other measures of risks. We remind that the skewness is a measure of the asymmetry of the probability distribution of a real-valued random variable. If the skewness is positive, the right tail of the distribution is longer than the left tail. The Kurtosis is a measure of the peakiness of the probability distribution. The Kurtosis of a normal random variable is equal to 3.

VI. VALUE AT RISK AND TAIL VAR

The Value at Risk is originally defined by the ε quantile of the wealth at the term of a predetermined period of time, T . This risk measure is used to appraise the shortfall risk and despite its drawbacks (see Artzner et al. (1999)), this remains today a key tool for asset allocation. If $\varepsilon \in [0, 1]$ is the maximum shortfall probability accepted by the investor, then the VaR is defined as the shortfall level, q , such that:

$$P(V_T \leq q) = \varepsilon. \quad (31)$$

Another common measure of risk is the TailVaR. The TailVaR, noted tv , is the expected shortfall level whether the value of the fund breaches the VaR q :

$$tv = E(V_T | V_T \leq q) \quad (32)$$

Before developing the numerical scheme used to compute the VaR and Tail VaR, we transform the eq. (31) and (32). According to the definition of V_t , the expression (31) is equivalent to

$$P(C_T \leq q - F(T)) = \varepsilon$$

If we remember eq. (17), the VaR level q also satisfies the equality

$$P\left(R_T \leq \ln\left(\frac{q - F_0 e^{rT}}{C_0} e^{-(r-mr)T}\right)\right) = \varepsilon \quad (33)$$

where R_T is defined as follows:

$$R_T = \ln\left(\frac{S_T}{S_0}\right)^m \quad (34)$$

$$= \sum_{i=1}^N \int_0^T m \left(\mu(i) - \frac{\sigma(i)^2}{2} \right) \delta(i, \alpha_u) du + \int_0^T m \sigma(i) \delta(i, \alpha_u) dW_u.$$

The TailVar can also be reformulated in term of R_T as follows:

$$tv = E\left(C_0 e^{(r-mr)T} e^{R_T} + F_0 e^{rT} | R_T \leq \ln\left(\frac{q - F_0 e^{rT}}{C_0} e^{-(r-mr)T}\right)\right) \quad (35)$$

If we denote by $f_{R_T}(s)$ the probability density function of R_T , eq (33) and (35) can respectively be rewritten as:

$$\int_{-\infty}^{\ln\left(\frac{q - F_0 e^{rT}}{C_0} e^{-(r-mr)T}\right)} s f_{R_T}(s) ds = \varepsilon \quad (36)$$

and

$$tv = \frac{1}{\varepsilon} \int_{-\infty}^{\ln\left(\frac{q - F_0 e^{rT}}{C_0} e^{-(r-mr)T}\right)} \left(C_0 e^{(r-mr)T} e^s + F_0 e^{rT}\right) f_{R_T}(s) ds \quad (37)$$

The probability distribution of R_T (conditionally to F_0) is not analytically determined. However, its characteristic function, which is also the Fourier Transform of its density has a closed form expression, detailed in the next proposition. By inverting numerically the characteristic function of R_T , it is hence possible to determine the density of probabilities of R_T and to compute numerically the VaR and Tail VaR by eq. (36) and (37). The numerical scheme will be detailed in the sequel of this section.

Proposition 1. The characteristic function of R_T , noted $\phi(v)$, is by definition the Fourier Transform of R_T :

$$\begin{aligned} \phi(v) &= E\left(e^{ivR_T}\right) \\ &= \int_{-\infty}^{+\infty} e^{ivs} f_{R_T}(s) ds \end{aligned}$$

where $f_{R_T}(s)$ is the density function of R_T . If we define the matrix B_{ivm} as

$$B_{ivm} = Q + \text{diag} \begin{pmatrix} ivm \left(\mu(1) - \frac{\sigma(1)^2}{2} \right) - \frac{1}{2} v^2 m^2 \sigma(1)^2 \\ M \\ ivm \left(\mu(N) - \frac{\sigma(N)^2}{2} \right) - \frac{1}{2} v^2 m^2 \sigma(N)^2 \end{pmatrix},$$

The characteristic function of R_T is equal to:

$$\phi(v) = \sum_{i=1}^N p_i(0) \langle \exp(B_{ivm}T) e_i; 1 \rangle$$

Proof. By definition of R_T , we have the relation:

$$\begin{aligned}\phi(v) &= E\left(e^{ivR_T}\right) \\ &= E\left[e^{\ln\left(\frac{S_T}{S_0}\right)^{ivm}}\right] \\ &= E\left[\left(\frac{S_T}{S_0}\right)^{ivm}\right],\end{aligned}$$

and this last expectation is calculated by the proposition 1.

The probability density function $f_{R_T}(s)$ can be retrieved by inverting the Fourier's transform $\phi(v)$:

$$\begin{aligned}f_{R_T}(s) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(v) e^{-ivs} dv \\ &= \frac{1}{\pi} \int_0^{+\infty} \phi(v) e^{-ivs} dv,\end{aligned}\quad (38)$$

where the second equality results from the symmetry of the integrand, which is itself due to the fact that the density function is real (no imaginary component).

The calculation of the integral (39) is done numerically by the Fast Fourier Transform algorithm (standard softwares such Matlab or Scilab propose functions inverting Fourier Transform). This method is quicker than computing the discretization of the integral eq. (38). The FFT algorithm computes in only $O(n \log n)$ operations (rather than $O(n^2)$ for other approaches), for any input array $\{IN(j): j = 0, \dots, N-1\}$, the following output array:

$$OUT(k) = \sum_{j=0}^{N-1} e^{-\frac{2\pi i k j}{N}} \cdot IN(j) \quad k = 0, \dots, N-1$$

The first step to use this numerical method, consists in discretizing the integral (39). We note Δv , the step of discretization and N , the number of steps. The mesh of discretization is defined as follows:

$$\{v_j\} = \{j \Delta v \in R^+ | 0 \leq j \leq N-1\}$$

One next defines a mesh of discretization for the values of $f_{R_T}(\cdot)$, spaced by Δs , and counting the same number N of elements as $\{v_j\}$ (this is a necessary condition to use the FFT algorithm):

$$\{s_k\} = \{-s_{min} + k \Delta s \in R^+ | 0 \leq k \leq N-1\}$$

where $s_{min} = \frac{N \Delta s}{2}$. On the condition that steps of discretization Δv and Δs satisfy the equality:

$$\Delta v \Delta s = \frac{2\pi}{N},$$

the discrete versions of equality (39), for all $s_{k=0 \dots N-1}$, can be reformulated into a suitable form for the FFT algorithm as follows:

$$\begin{aligned}\underbrace{\pi f_{R_T}(s_k)}_{OUT(k)} &= \int_0^{+\infty} \phi(v) e^{-ivs_k} dv \\ &= \sum_{j=0}^{N-1} e^{-i \frac{2\pi}{N} j k} \underbrace{e^{i v_j \cdot s_{min}} \phi(v_j) \Delta v}_{IN(j)},\end{aligned}\quad (39)$$

The left hand term of equation (39) is the output vector computed by a standard FFT algorithm. Once that the density function is estimated, one can easily infer the ε -quantile of R_T , that we note r_ε and infer from formula (36) the VaR level q :

$$q = \ln\left(C_0 e^{(r-mr)T} e^{r_\varepsilon} + F_0 e^{rT}\right)\quad (40)$$

The TailVaR can be computed by discretizing the integral (37):

$$tv = \frac{1}{\varepsilon} \sum_{s_k = \ln\left(\frac{q - F_0 e^{rT}}{C_0} e^{-(r-mr)T}\right)} \left(C_0 e^{(r-mr)T} e^{s_k} + F_0 e^{rT}\right) f_{R_T}(s_k) \Delta s\quad (41)$$

VII. APPLICATIONS

Based upon the calibration procedure of section 4, we have fitted a three states regime switching model to 1771 daily returns of the CAC40 (the French main stocks index) observed on the period running from the 2/1/2003 to the 30/11/2009. The parameters obtained by MLE are presented in table 1. The log-likelihood is worth 5397.

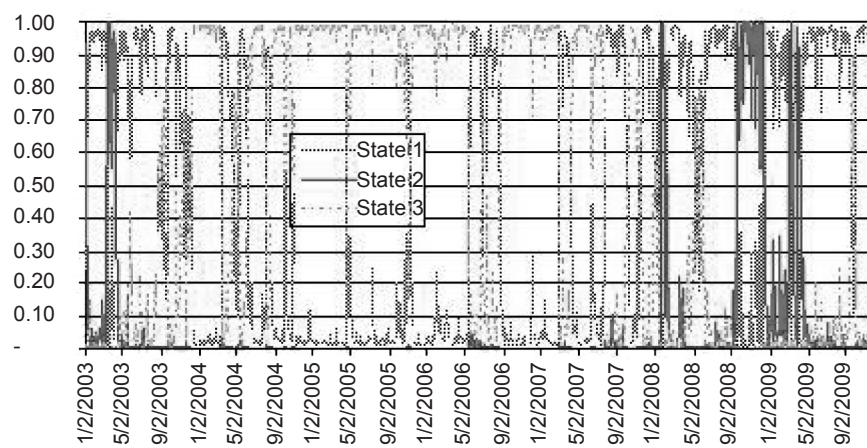
Table 1. Parameters

Parameters	Values (MLE), daily basis
$\mu(1)$	-0.05%
$\mu(2)$	-0.10%
$\mu(3)$	0.09%
$\sigma(1)$	1.45%
$\sigma(2)$	3.81%
$\sigma(3)$	0.72%

One directly observes that the first state corresponds to a bear market, characterized by a negative average and a high volatility of returns. The second state corresponds to a situation of crisis, characterized by a high negative average and a very high volatility of returns. On the opposite, in state 3 the daily return has a positive mean and a lower volatility than in state 1 and 2. This state is clearly linked to a period of economic growth. The matrix of transition probabilities, on a period of one day takes the values:

$$p(t, t+1 \text{ days}) = \begin{pmatrix} 0.9839 & 0.0070 & 0.0091 \\ 0.0480 & 0.9520 & 0 \\ 0.0096 & 0 & 0.9904 \end{pmatrix}$$

Figure 1. Probabilities of presences in states 1, 2 and 3



One sees that the probabilities of remaining in state 1, 2 and 3 are high. We observe a first important feature: the probabilities to transit from state 3 to 2 or from state 2 to 3 are null. This means that the financial system cannot jump from a bull market to a state of crisis without transiting by a moderate bear market.

Figure 1. presents the probabilities of presence in states 1 2 or 3, conditionally to observed returns (the $p_i(t_k | \Theta, O_1, \dots, O_k)$, such as defined by eq. (13)). This reveals that the probability that the CAC 40 was in state 2, is high from the 22/9/2008 till the 11/12/2008. This period corresponds to the heart of the recent liquidity crisis.

We have considered CPPI funds having an initial value of 100 millions and tracking the CAC40 index. The floor F_0 is set to 80 millions. The chosen multipliers, m , are worth {0.25 0.50 0.75 1.00 1.25 1.50}. The risk free rate r is set to 1.5%. For a time horizon of one year, the first four moments of those funds are presented in table 2. The higher is the multiplier, the higher are

the expected returns and volatilities. However, we note that the performance of CPPI funds is relatively close to the risk free rate and even below for multipliers m smaller than 0.75. To understand this interesting observation, we need to remind that the multiplier may also be seen as the speed at which we purchase or sell stocks in a rising or declining market. So, if $m \leq 0.75$, the speed at which we purchase stocks in state 3 (the bull market) is not sufficient to profit of the upward leverage effect before jumping to states 1 and 2. On the opposite situation (a bear market or a severe crisis), if $m \leq 0.75$, we don't sell sufficiently quickly the stocks hold in portfolio and suffer from a loss.

The volatility and the Kurtosis increase with the multiplier while the Skewness follows an opposite trend. We remind the reader that the Skewness and the Kurtosis of a Gaussian random variable take respectively a value of 0 and 3.

The probability density functions of fund market values, after one year, can be retrieved by inverting a

Table 2. Moments

Multiplier m	$E(V_t)$	$V(V_t)$	$S(V_t)$	$K(V_t)$
0.25	101.45	1.21	9431.53	3.06
0.50	101.47	2.42	1170.29	3.23
0.75	101.55	3.66	342.69	3.53
1.00	101.71	4.95	142.43	3.98
1.25	101.95	6.31	71.76	4.62
1.50	102.26	7.77	40.94	5.47

Figure 2: Probability density of a CPPI fund

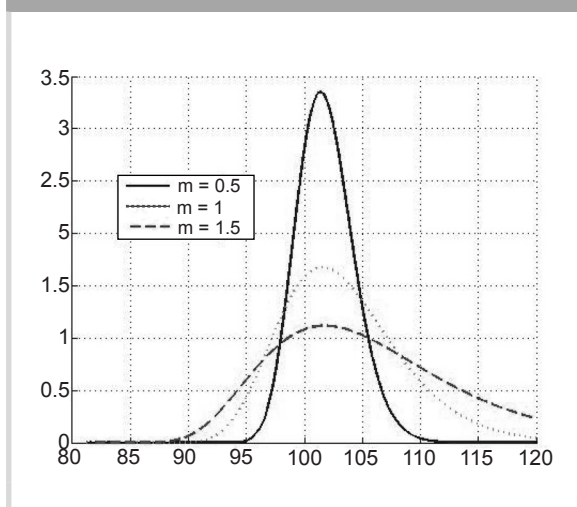
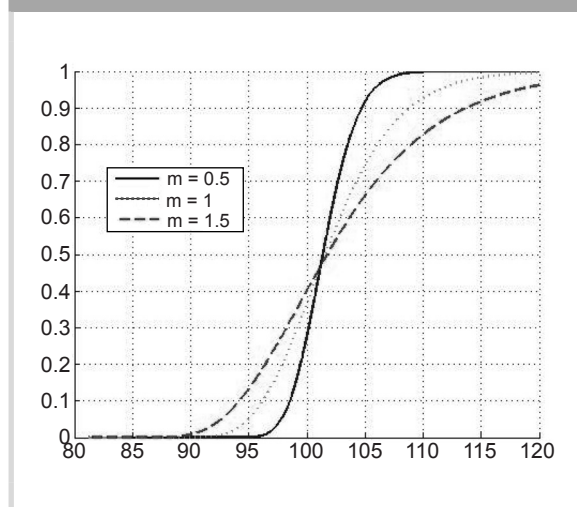


Figure 3: Probability distributions of a CPPI fund



Fourier transform. The step of discretization and the number of steps, involved in the FFT algorithm are set to $\Delta v = 0.5$ and $N = 2^{14}$. The size of the interval ΔS is hence equal to $2\pi / (N\Delta v) = 7.66e - 4$. The figure 2 plots the density functions of $V_{T=1\text{year}}$. The multiplier acts as a leverage on the breadth of the density. The higher is the multiplier, the wider is the density. The corresponding cumulative density functions are plotted in the figure 3.

The table 3 contains the VaR and Tail VaR for a confidence level of 5% and 1%. Raising the multiplier decreases the VaR and the Tail VaR. If the minimum capital required to hedge eventual future losses of the fund is valued as the difference between the initial value of the fund and the 1 year VaR, the capital for $m = 1.5$ must be respectively above 8.38 or 9.74 million for a confidence level of 5% and 1%. If the minimum capital is defined as the difference between the initial value of the fund and the 1 year Tail VaR, the capital must be respectively above 9.16 or 10.94 million for a confidence level of 5% and 1%.

VIII. CONCLUSIONS.

This work explores some features of funds managed by CPPI, when a switching regime process drives the underlying risky asset. The interest of working with such kind of processes is clear: the recent crisis has confirmed that the average trend and volatility of any risky asset can jump from one set of value to another one. Under the standard assumptions of frictionless markets, we have established the first fourth centered moments of a CPPI fund, in a continuous time framework. Next, we have detailed a procedure to infer from the Fourier Transform of $\ln\left(\frac{S_T}{S_0}\right)^m$, the density function by a standard FFT algorithm. Based upon those results, a method to compute the VaR and TailVaR has been proposed.

In the last section of this work, we have fitted a 3 states switching regime model to daily return of the CAC 40 (the French main stocks index). We have observed that the states correspond respectively to a bull market, to a bear market and to a deep crisis. Furthermore, the

Table 3. Var and TailVar

Multiplier m	VaR 5%	TailVaR 5%	VaR 1%	TailVaR 1%
0.25	99.4081	96.7063	98.6831	95.3735
0.50	97.7937	96.7032	96.4987	95.2448
0.75	96.3109	94.7494	92.9060	92.8999
1.00	94.9607	93.1045	92.8523	91.7870
1.25	93.7409	92.1746	91.4359	90.0236
1.50	92.6204	90.8361	90.1575	89.0766

probabilities of transition between states of growth and of crisis are null. This means that the financial system cannot jump from a bull market to a state of crisis without transiting by a moderate bear market. We have next considered CPPI funds tracking the CAC40 index. As expected, the one year average return is proportional to the CPPI multiplier. But, when this multiplier is too small, the return can fall below the risk free rate. To understand this point, we need to

remind that the multiplier may also be seen as the speed at which we purchase or sell stocks in a rising or declining market. So, if the multiplier is not sufficient, we don't sell sufficiently quickly the stocks hold in portfolio in declining markets and suffer regularly from a loss. ■

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Appendix

In this appendix, we sketch the proof of the result of Buffington and Elliott (2002) stating that:

$$E\left(e^{i\sum_{i=1}^{N-1}\omega_i T_i(t,u)} | G_t \vee H_t\right) = \left\langle \exp\left(\left(Q' + i \operatorname{diag}(\omega_1, \omega_2, \dots, \omega_{N-1}, 0)\right)(u-t)\delta(t); 1\right) \right\rangle \quad (42)$$

where $\delta(t) = (\delta(i, \alpha(t)))_{i \in N}$ is a vector taking its values in the set of units vectors $\{e_1, e_2, \dots, e_N\}$. From the theory of Markov processes, we know that the vector $\delta(u)$ may be rewritten as:

$$\delta(u) = \delta(t) + \int_t^u Q' \delta(s) ds + \int_t^u dM_s, \quad (43)$$

where M_s is a martingale. Let us define the R^N valued process Z_t as follows:

$$Z_u = \exp\left(i \int_0^u \langle \omega, \delta(s) \rangle ds\right) \delta(u),$$

where ω is the vector $(\omega_1, \omega_2, \dots, \omega_{N-1}, 0)$. Given eq. (43), the differential of Z_u is equal to

$$\begin{aligned} dZ_u &= \exp\left(i \int_0^u \langle \omega, \delta(s) \rangle ds\right) d\delta(u) + \exp\left(i \int_0^u \langle \omega, \delta(s) \rangle ds\right) i \langle \omega, \delta(u) \rangle \delta(u) du \\ &= \left(Q' + i \operatorname{diag}(\omega)\right) Z_u du + \exp\left(i \int_0^u \langle \omega, \delta(s) \rangle ds\right) dM_u, \end{aligned}$$

and therefore

$$Z_u = \delta(t) + \int_t^u \left(Q' + i \operatorname{diag}(\omega)\right) Z_s ds + \int_t^u \exp\left(i \int_t^v \langle \omega, \delta(s) \rangle ds\right) dM_v$$

The integral with respect to M_v being a martingale, the expectation of Z_u is solution of:

$$E\left(Z_u | G_t \vee H_t\right) = \delta(t) + \int_t^u \left(Q' + i \operatorname{diag}(\omega)\right) E\left(Z_s | G_t \vee H_t\right) ds,$$

which is

$$E\left(Z_u | G_t \vee H_t\right) = \exp\left(\left(Q' + i \operatorname{diag}(\omega)\right)(u-t)\delta(t)\right). \quad (44)$$

The characteristic function of $\sum_{i=1}^{N-1} \omega_i T_i(t, u)$ may be restated as:

$$E\left(e^{i\sum_{i=1}^{N-1}\omega_i T_i(t,u)} | G_t \vee H_t\right) = E\left(e^{i \int_t^u \langle \omega, \delta(s) \rangle ds} | G_t \vee H_t\right).$$

The relation (42) is a consequence of the next equality and of eq. (44):

$$\begin{aligned} E\left(e^{i \int_t^u \langle \omega, \delta(s) \rangle ds} | G_t \vee H_t\right) &= E\left(\left\langle e^{i \int_t^u \langle \omega, \delta(s) \rangle ds} \delta(u); 1 \right\rangle | G_t \vee H_t\right) \\ &= \left\langle \exp\left(\left(Q' + i \operatorname{diag}(\omega)\right)(u-t)\delta(t); 1\right) \right\rangle \end{aligned}$$

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