Hedge Fund Returns and Factor Models:
A Cross-Sectional Approach*

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I. Introduction

In the last two decades, interest in hedge funds from both academics and investors has grown dramatically. These funds are typically organized as private investment vehicles for wealthy individuals and institutional investors. Since they do not have to disclose their activities publicly, little is known about the risk in hedge fund strategies. The lack of transparency and the fear for style drift have raised the question whether it is possible to identify and estimate the risk factors driving hedge fund returns. Factor models are employed to capture their main characteristics and thus identify the risk exposures.

These models are initially developed to explain common risk affecting equity returns. They are a natural extension of the one-factor CAPM [Sharpe (1964), Lintner (1965) and Black (1972)] and use a set of observed variables, such as market indexes and economic indicators, as proxies for common risk factors. The growth of the hedge fund industry has reoriented the asset pricing efforts toward alternative returns offered by hedge funds. For example, Fung and Hsieh (2004) showed that equity-oriented hedge fund indexes have two major exposures: the equity market as a whole, and the spread between small cap and large cap stocks[1].

An extensive literature has documented that hedge fund returns differ from those of the traditional assets. Mutual funds returns have high and positive correlation with asset class returns, which suggests that they behave as a «buy and hold» strategy. Hedge fund returns seem to have low and sometimes negative correlation with traditional asset class returns[2]. This suggests that they behave as if deploying a dynamic strategy including short sells and leverage. Nonlinear payoffs and time-varying exposures to risk factors are some resulting stylized facts [see Fung and Hsieh (1997a, 1997b, 2001, 2002a), Mitchell and Pulvino (2001), and Agarwal and Naik (2001, 2004) among others]. It follows that to analyze hedge fund returns, one has to take into account that their risk exposures are nonlinear and likely to change very frequently. For example, Agarwal and Naik (2004) introduce option-based factors to capture nonlinear payoffs of hedge fund strategies and show that left-tail risk is a priced factor. Fung and Hsieh (1997a) focus on the time-varying risk exposures of hedge funds and show that the Trend Follower index returns are positively correlated with the stock market in situations of bullish markets and negatively correlated in bear markets. Hasanhobvic and Lo (2007) use observed factors, such as the S&P 500 index, the USD return index, the Bond Index, etc., to model the returns of individual hedge funds[3]. They estimate the risk exposures using a 24-month rolling window. This methodology has one main drawback: the factor model specification is determined in advance and is kept unchanged through the entire sample period[4]. This factor selection mechanism[5] does not take into account the time-varying risk profile of hedge fund returns. Therefore, hedge fund analysis should consider that, for different rolling periods, a given investment strategy may not be exposed to the same risk factors.

The factor selection problem is not new in the literature. The main issue is related to the delicate balance between using too many or too few factors. Specifically, adding too many factors lowers the regressors efficiency. Working with too few factors also has an important hidden cost, the model risk. This raises the question whether it is possible to build a factor selection methodology allowing to consider only the appropriate factors. In this paper, we develop a dynamic factor-based approach to explain hedge fund returns. First, we focus on an approximate factor model framework to deal with the factor selection issue. Instead of determining in advance which factors to include in the analysis, we use asymptotic developments of Bai and Ng (2002, 2006) to select the relevant factors. We estimate the risk dimension, *i.e.* the optimal number of latent factors, using individual hedge fund returns. Then, we asses the economic interpretation of these factors by matching them with the observed

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variables. We thus identify which economic forces drive hedge fund returns. Second, we take into account the instability of asset risk profile by using rolling period analysis to estimate time-varying hedge fund risk exposures. Individual hedge fund returns are used instead of index returns. This choice allows us to go further in the comprehension of the latent factor structure. The information on the common behavior of fund returns is filtered not only from the past historical data (time-series dimension), but also from the cross-section of fund returns. The asymptotic tests we perform hereafter are consistent for large cross-section and moderately large time-series dimension. This data configuration is clearly more in line with the dynamic factor selection objective we aim at. Moreover, Chan and al. (2005) point out that a disaggregated approach may yield additional insights not apparent from index-based risk models.

Finally, we use the HFR database to test our approach on a set of individual equity hedge funds. One possible application of risk factor models is hedge fund replication. This is a challenge that has naturally appeared in response to drawbacks inherent to these funds such as opacity, lack of liquidity and high incentive fees. Repliquants generate hedge-fund-like returns using more liquid assets such as market indexes. This allows to construct benchmarks and then get a better evaluation of alpha generation for a given fund. We use hedge fund replication as a criterion for assessing the quality of the dynamic factor-based approach developed herein, as well as the benefits of dynamic factor selection mechanism. We find that the hedge fund clone index constructed by our methodology outperforms the «naive» clone index constructed by a methodology consisting of a *static* and *ad hoc* factor selection, as in Hasanhobvic and Lo (2007).

The paper is organized as follows. In section II, we propose a statistic model for large panel data and describe how recent asymptotic tests can be used to assess the common risk structure of asset returns. Section III deals with the economic interpretation of the latent factors. Section IV describes our dynamic factor-based approach to analyze hedge fund returns and discuss the empirical results. Section V concludes.

■ II. Determining the number of latent factors

In this section, we focus on the approximate factor model framework and use Bai and Ng (2002) asymptotic tests to estimate the risk dimension.

II.1. AN APPROXIMATE FACTOR MODEL FOR HEDGE FUND RETURNS

When dealing with large panel data, the arbitrage pricing theory (APT) of Ross (1976) assumes that a small number of factors can be used to explain a large number of asset returns. Let X_{it} be the observed return for fund i at time t, for $i = 1,...,N$ and $t = 1,...,T$. We consider the following model with r common factors:

$$X_{it} = \lambda_i' F_t + e_{it}, \qquad (1)$$

where F_t is a $r \times 1$ vector of common factors, λ_i is a $r \times 1$ vector of factor loadings for the fund i, and e_{it} is the i th element of the t th column of the idiosyncratic component matrix. $\lambda_i' F_t$ represents the common component of X_{it}. The idiosyncratic components are supposed to have zero mean.

We place our analysis in an approximate factor model framework in the sense of Bai and Ng (2002) which is more realistic in hedge fund world, since it allows for weak time and cross-section dependence and heteroscedasticity in the idiosyncratic components. The factors, their loadings, as well as the idiosyncratic errors are not observable and have to be estimated. Although it seems appealing to assume one factor, there is growing evidence against the adequacy of a single factor model in explaining hedge fund returns. For example, Fung and Hsieh (1997a, 1997b) show that hedge fund risk exposures are multidimensional and highly dynamic. Thus, instead of restricting the analysis by fixing $r = 1$, we propose a procedure to determine the appropriate number of factors[6].

Determining the number of factors in approximate factor models is an important issue when dealing with large panel data in both cross-sectional (N) and time-series (T) dimension. In classical factor analysis (see Anderson (1984)), N is assumed fixed, the factors are independent of errors e_t and the covariance matrix of the idiosyncratic components Σ is diagonal. Under these assumptions, a root-T consistent and asymptotically normal estimator of Σ, as measured by the sample covariance matrix, can be obtained. The essential of classical factor analysis applies to the case of large N but fixed T since the $N \times N$ problem can be reformulated as a $T \times T$ problem, as discussed by Connor and Korajczyk (1993) among others.

Inference on r under classical assumptions is based on the eigenvalues of the estimator of Σ. Indeed, a characteristic of a panel of data generated from r factors is that the first r largest eigenvalues of the $N \times N$ covariance matrix of X_t diverge as N increases to infinity but the $(r+1)$ th eigenvalue is bounded (see for example Chamberlain and Rothschild (1983)). However, it can be shown that all nonzero eigenvalues of $\widehat{\Sigma}$ (not just the first r) increase with N, and a test based on the sample eigenvalues is thus not viable. A likelihood ratio test can also, in theory, be used to estimate the number of factors under the assumption that e_{it} is normally distributed. But as discussed by Dhrymes et al. (1984), the number of statistically significant factors estimated by the likelihood test ratio increases with N even if the true number of factors is fixed. Connor and Korajczyk (1993) develop a test for the number of factors in the asset returns, which is derived under sequential limit assumptions, *i.e.* N converges to infinity with a fixed T, then T converges to infinity. In addition, covariance stationarity and homoscedasticity are crucial for the validity of their test. As discussed by Bai and Ng (2002), the fundamental problem in classical analysis is that the theory does not apply when both N and T go to infinity. This is because consistent estimation of Σ (whether it is an $N \times N$ or $T \times T$ matrix) is not a well defined problem[7].

To address this issue, Bai and Ng (2002) develop an asymptotic theory for factor models with large panel data (when N, $T \to \infty$). Their analysis is not standard since: i) the sample size in both the cross-section and time-series dimension have to be taken into account; ii) the factors are not observed. Based on an approximate factor model framework, they first establish the convergence rate for the factor estimates that will allow for consistent estimation of r. They then propose some panel criteria and show that the number of factors can consistently be estimated. Inference on r is set up as a model selection problem and the proposed criteria depend on the trade-off between good fit and parsimony. The penalty for overfitting is a function of both N and T in order to consistently estimate the number of factors. Consequently, the usual AIC and BIC criteria, which are functions of N or T alone, do not work when both dimensions of the panel are large. In addition, their theory holds under heteroscedasticity and weak cross-section and serial dependance in the idiosyncratic components. These additional assumptions are reported in the appendix A. Note that Bai and Ng (2002) approximate factor structure is more general than that of Chamberlain and Rothschild (1983), which focuses only on the cross-section behavior of the data by allowing for weak cross-section dependence.

II.2. ASYMPTOTIC TESTS FOR ESTIMATING THE NUMBER OF LATENT FACTORS

Equation (1) can be written in a more general way as:

$$X = F\Lambda' + e, \qquad (2)$$

where X is the $T \times N$ matrix of individual hedge fund returns, e is the T x N matrix of idiosyncratic components, Λ is the $N \times r$ matrix of factor loadings, and F is the $T \times r$ matrix of common factors.

We use the asymptotic principal component method to estimate the factors and their loadings. This method minimizes the following objective function:

$$V(k) = \min_{\Lambda, F^k} (NT)^{-1} \sum_{i=1}^{N} \sum_{t=1}^{T} (X_{it} - \lambda_i^k F_t^k)^2, \qquad (3)$$

subject to the normalization of either $(\Lambda^k)'\Lambda^k / N = I_k$ or $(F^k)'F^k / T = I_k$. The superscript in λ_i^k and F_t^k signifies the allowance of k factors in the estimation, with $k = min(T,N)$. There are two possible solutions.

i) Concentrating out F^k and using the normalization $\Lambda^{k'}\Lambda^k / N = I_k$, the estimated factor loading matrix $\overline{\Lambda^k}$ is $\sqrt{N}$ times the eigenvectors corresponding to the k largest eigenvalues of the $N \times N$ covariance matrix $X'X$. Given $\overline{\Lambda^k}$, $\overline{F^k} = X\overline{\Lambda^k} / N$ represents the corresponding matrix of the estimated common factors.

ii) The second is given by $(\tilde{F}^k, \tilde{\Lambda}^k)$, where $\tilde{F}^k$ represents $\sqrt{T}$ times the eigenvectors corresponding to the k largest eigenvalues of the $T \times T$ covariance matrix XX'. The normalization that $(F^k)'F^k / T = I_k$ implies that $\tilde{\Lambda}^{k'} = (\tilde{F}^k)'X / T$ is the corresponding matrix of the estimated common factors.

The first solution is less costly when $T > N$, while the second is more appropriate when $T < N$. As it will be discussed later, our dynamic approach uses T-month rolling period estimations, with T smaller than the number of individual hedge funds N. We thus retain the second set of principal component calculations to estimate the factors and their loadings.

Bai and Ng (2002) propose some criteria to estimate the number of factors r. Let F^k be a matrix of k factors and

$$V(k, F^k) = \min_{\Lambda} \frac{1}{NT} \sum_{i=1}^{N} \sum_{t=1}^{T} (X_{it} - \lambda_i^{k'} F_t^k)^2, \qquad (4)$$

be the sum of squared residuals when k factors are estimated. Then a loss function $V(k, F^k) + kg(N,T)$, with $g(N,T)$ being the penalty for overfitting, can be used to determine k. The authors propose some penalty functions $g(N,T)$ such as the following criteria of form $IC(k) = V(k, F^k) + kg(N,T)$ can consistently estimate r:

$$\begin{aligned} IC_1(k) &= \ln(V(k, \tilde{F}^k)) + k(\frac{N+T}{NT})\ln(\frac{NT}{N+T}), \\ IC_2(k) &= \ln(V(k, \tilde{F}^k)) + k(\frac{N+T}{NT})\ln C_{NT}^2, \\ IC_3(k) &= \ln(V(k, \tilde{F}^k)) + k(\frac{\ln C_{NT}^2}{C_{NT}^2}). \end{aligned} \qquad (5)$$

In these equations, $C_{NT} = \min(\sqrt{N}, \sqrt{T})$, $\tilde{F}$ is the matrix of estimated common factors and $V(k, \tilde{F}^k) = N^{-1} \sum_{i=1}^{N} \widehat{\sigma_i}^2$, $\widehat{\sigma_i}^2 = \tilde{e}_i \tilde{e}_i / T$. The IC_1, IC_2 and IC_3 criteria are called Information Criteria[8].

We use Monte Carlo simulations to assess the finite sample properties of IC criteria relative to our data configuration. The simulation procedure is described in the Appendix B. The results reported in Table 3 show that IC_2 outperform the other criteria in inferring the appropriate number of factors. Moreover, a small degree of correlation in the idiosyncratic errors does not affect their finite sample properties.

III. Economic interpretation of risk exposures

Based on the Bai and Ng (2006) framework, we discuss how to match the latent factors to observed variables. This step is essential to identify which economic forces drive hedge fund returns.

III.1. MATCHING OBSERVED VARIABLES TO LATENT COMMON FACTORS

The factors represent the common shocks that drive the covariation of asset returns. These common factors are directly determined by the covariance structure of the data[9]. It is therefore intuitive to replace the unobserved factors with statistically estimated ones. For example, Lehman and Modest (1988) use factor analysis, while Connor and Korajzcyk (1986, 1988) adopt the method of principal components. However, these statistic factors do not have direct economic interpretation.

Another possibility is to select a set of observed variables as proxies of the unobserved latent factors. For example, in the CAPM analysis equity index returns are used as proxies of the unobserved market portfolio returns. Chen et al. (1986) find that the factors in the APT are related to macroeconomic variables. Fama and French (1993) propose three well-known observed factors: the market excess return (MKT), the small minus big (SMB), and the high minus low (HML) factors. Later on, Carhart (1997) extends the three factor model of Fama and French (1993) by adding a forth factor in order to take into account risk related to return persistence. Agarwal and Naik (2004) use option-based factors to account for nonlinear returns generated by dynamic investment strategies employed by hedge funds.

There is a certain appeal in associating the latent factors with the observed variables in order to facilitate the economic interpretation of the common variations of asset returns. However, as pointed out by Shanken (1992), estimation of betas using proxy factors is relevant only if the fundamental factors are spanned by the observed variables. Such a condition is breached even if a pure measurement error is added to a perfect proxy.

Suppose we observe an $m \times 1$ vector of economic variables, denoted G_t. We want to figure out whether its elements are generated by (or are linear combinations of) the r latent factors F_t. As discussed in Bai and Ng (2006), considering G_{jt} to be an exact linear combination of the latent factors is a rather strong assumption. An observed series might explain the variations of the latent factors very closely, and yet is not an exact factor in a mathematical sense. This is due, for example, to measurement errors and time aggregation (see, for example, Breeden et al. (1989)). For that reason, we consider an approximate relation between the observed and the latent factors:

$$G_{jt} = \delta' F_t + \varepsilon_{jt}, \tag{6}$$

where F_t is a $r \times 1$ vector of latent factors and $\varepsilon_{jt} \sim N(0, \sigma_\varepsilon^2(j))$.

Bai and Ng (2006) show that the space spanned by the latent factors can consistently be estimated when the simple size is large in both the cross-section and the time series dimensions[10]. They develop some criteria to match the observed variables with the estimated latent factors.

III.2. ASYMPTOTIC TESTS

Suppose that we observe G_{jt}, $j = 1, \ldots, m$ and $t = 1, \ldots, T$. We want to test if it is generated by (or is a linear combination of) r latent factors. The latent factors F and their number r are not observed and have to be estimated.

We denote $\widehat{G}_{jt} = \widehat{\gamma}'_j \widetilde{F}_t$, where $\widetilde{F}_t$ is the principal component estimation[11] of F, $\widehat{\gamma}_j$ is obtained by least squares from a regression of G_{jt} on $\widetilde{F}_t$, and $\widehat{\varepsilon}_{jt} = G_{jt} - \widehat{G}_{jt}$. The residuals $\widehat{\varepsilon}_{jt}$ are referred as a measurement error, even though it might be due to systematic differences between F_t and G_{jt}.

We consider two statistics proposed by Bai and Ng (2006) to compare the observed variables with the estimated factors $\widetilde{F}$.

$$NS(j) = \frac{\widehat{var}(\widehat{\varepsilon}(j))}{\widehat{var}(\widehat{G}(j))}, \tag{7}$$

$$R^2(j) = \frac{\widehat{var}(\widehat{G}(j))}{\widehat{var}(G(j))}, \tag{8}$$

where a consistent estimate of $\widehat{var}(\widehat{G}_{jt})$ is given by:

$$\frac{1}{N} \widehat{\gamma}'_j \widetilde{V}^{-1} \widetilde{\Gamma}_t \widetilde{V}^{-1} \widehat{\gamma}_j. \tag{9}$$

In this equation, $\widetilde{V}$ is a $\widetilde{r}$ x $\widetilde{r}$ diagonal matrix consisting of the $\widetilde{r}$ largest eigenvalues of sample covariance matrix XX' / N, and $\widetilde{\Gamma}_t$ is a consistent estimate of $\Gamma_t = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} E(\lambda_i \lambda'_j e_{it} e_{jt})$. To allow for heteroskedastic errors e_{it}, $\widetilde{\Gamma}_t$ is given as follows:

$$\widetilde{\Gamma}_t = \frac{1}{N} \sum_{i=1}^{N} \widetilde{e}_{it}^{2} \widetilde{\lambda}_i \widetilde{\lambda}'_i, \tag{10}$$

where $\widetilde{\lambda}_i$ and $\widetilde{e}_{it}$ are respectively the factor loadings and the idiosyncratic errors resulting from the principal component computations as described in section 2.2.

The statistics given in equations (7) and (8) consider one observed variable G_j at a time. The $NS(j)$ statistic represents a noise-to-signal ratio. If G_j is an exact factor, *i.e.* $G_{jt} = \delta' F_t$ the population value of $NS(j)$ is zero. Concerning $R^2(j)$, the higher this statistic, the higher the adequacy between the observed and the latent factors.

Finally, we use Monte Carlo simulations to assess the finite sample properties of the proposed criteria, as well as their critical values. Simulation procedure is described in the Appendix C. The results reported in Table 5 show that both criteria help identify the appropriate observed factors. In the following section we show how we use these criteria to select the relevant observed factors and apply this factor selection procedure to a set of individual equity hedge funds.

■ IV. Empirical applications

In this section, we develop a dynamic factor-based approach to analyze equity hedge fund returns. Subsection 1 describes the data. In subsection 2, we estimate the risk dimension. Subsection 3 provides an economic interpretation of the covariance structure of fund returns, while subsection 4 presents the dynamic hedge fund return analysis and discusses the empirical results.

IV.1. THE DATA

We use in all this section the HFR database providing returns of individual hedge funds. This database contains only the funds that are still "alive", *i.e.* active as of the end of our sample period, December 2005. We acknowledge that the database suffers from the survivorship bias. However, the importance of such a bias for our application is tempered by the fact that many successful funds leave the sample as well as the poor performers, reducing the upward bias in expected returns. In particular, Fung and Hsieh (2000) estimate the magnitude of survivorship bias to be 3% per year, and Liang's (2000) estimate is 2.24% per year. In addition, the focus of our study is on the relative performance of our dynamic approach versus a naive one which consists in including in the analysis all the available observed factors and keeping this set unchanged. It follows that, any survivorship bias should impact both approaches in the same way, leaving their relative performances unaffected. HFR database classifies funds into one of 17 different investment styles, listed in Table 6 in the Appendix D.

We limit our analysis to the individual funds of the equity hedge strategy for two reasons. First, this strategy involves quite homogenous equity-oriented funds investing on both the long and the short sides of the market. Thus, we expect the equity hedge funds to be more sensible to equity-based risk factors. Second, the number of funds N with full set of data for our studying period (January 1997 to December 2005) is large, which will improve the finite sample properties of our tests. We drop funds that: *i)* do not report net-of-fee returns; *ii)* report returns in currencies other than the U.S. dollar; *iii)* report returns less frequently than monthly; *iv)* have less than 10 Million US dollars of assets under management (AUM). These filters yield a final sample of 680 equity hedge funds.

IV.2. ESTIMATING THE NUMBER OF LATENT FACTORS THAT DRIVE EQUITY HEDGE FUND RETURNS

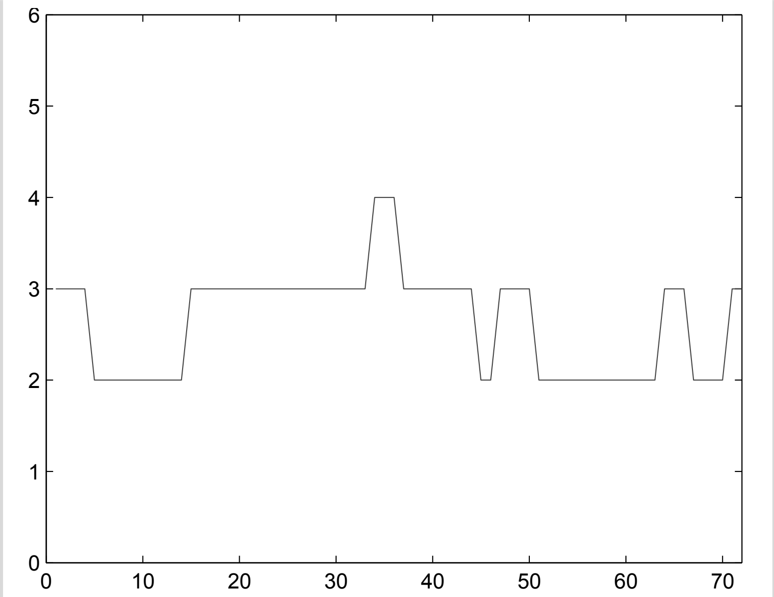
As discussed in subsection II.2, Monte Carlo simulation results (see the Appendix B) motivate the use of IC_2 criterion to estimate the number of latent factors. Since our study focuses on a dynamic approach, we have to choose the minimal length of the rolling window (T) ensuring good finite sample properties for the estimated parameters. T depends on the trade-off between a high dynamics of our approach and good finite sample properties of the Bai and Ng (2002) criteria, in particular IC_2. *i)* If T is too small the convergence is not achieved and the selected criteria will not yield good estimates of the number of latent factors. *ii)* If T is too large, our approach will lose its dynamic character.

The results reported in Table 3 in the Appendix B show that for $T = 24$ the finite sample properties of the six criteria are less precise than for $T = 36$. For instance, for $T = 24$ and $r = 3$ the IC criteria underestimate r. For $T = 36$, the IC criteria, and in particular IC_2, yield the appropriate number of factors more precisely. We use this choice in all the following empirical applications.

The length of the entire sample allows us to form 72 rolling windows of length 37 months for each one. The first rolling window goes from January 1997 to January 2000, the second from February 1997 to February 2000, ..., the last one extends from December 2002 to December 2005. We use the first 36 months of each rolling window to perform the factor selection procedure and to estimate the beta coefficients, while the last (the $37th$) observation is meant to compare hedge fund returns and model predictions[12].

Hedge fund returns are standardized previously within the 36 first months of each rolling period. Let X be the T by N matrix of the equity hedge fund returns of our sample data such that the i th column is the time series of fund i. Let $\widetilde{V}$ be a $r \times r$ diagonal matrix consisting of the r largest eigenvalues of XX' / NT. Let $\widetilde{F} = (\widetilde{F}_1, ..., \widetilde{F}_T)'$ be the principal component estimates of F under the normalization that $\frac{F'F}{T} = I_r$. Then $\widetilde{F}$ is comprised of the r eigenvectors (multiplied by $\sqrt{T}$) associated with the r largest eigenvalues of the matrix XX' / NT in the decreasing order. Let $\Lambda = (\lambda_1, ..., \lambda_N)'$ be the matrix of factor loadings. The principal component estimator of Λ is $\widetilde{\Lambda} = X' \widetilde{F} / T$ and $\widetilde{e}_{it} = X_{it} - \widetilde{\lambda}_i ' \widetilde{F}_t$.

Figure 1: The number of latent factors estimated using IC_2 criterion for each rolling period



Using the principal component estimates, we calculate the IC_2 criterion to estimate the number of latent factors. Figure 1 plots, for each rolling window, the estimated number of factors, which varies between 2 and 4. This result seems to be quite realistic for this kind of strategy and highlights the CAPM shortcomings when explaining hedge fund returns, even for an equity-oriented strategy.

Figures 2 and 3 plot the two first estimated latent factor and the S&P 500 index returns. $F1$ and $F2$ are the estimated latent factors corresponding to respectively the first and the second largest eigenvalue of the covariance matrix of the fund returns for the last rolling period which extends from December 2002 to December 2005. Since $F1$ and $F2$ are estimated using standardized data, we renormalize them in order to obtain the same standard deviation as for S&P 500 index. S&P 500 index returns have been centered by their mean in order to facilitate the comparison with the estimated latent factors. The correlation coefficients between the two latent factors with the S&P 500 index are respectively $0,85$ and $0,27$. The first factor behaves closely with the S&P 500 index, while the second one is less correlated with the equity market. Even if the equity market factor seems to play an important role in explaining the cross-section of equity hedge fund returns, we are yet unable to identify a significant portion of common risk represented by the second latent factor if we use a single factor model.

Figure 2: The first estimated latent factor F1 and the S&P 500 index returns from December 2002 to December 2005

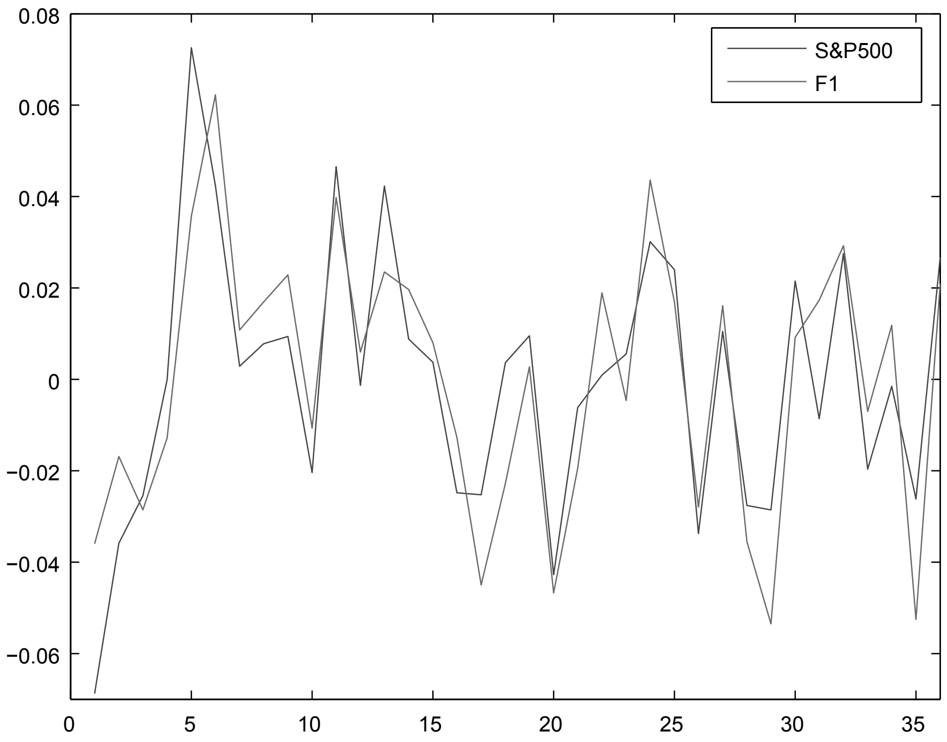
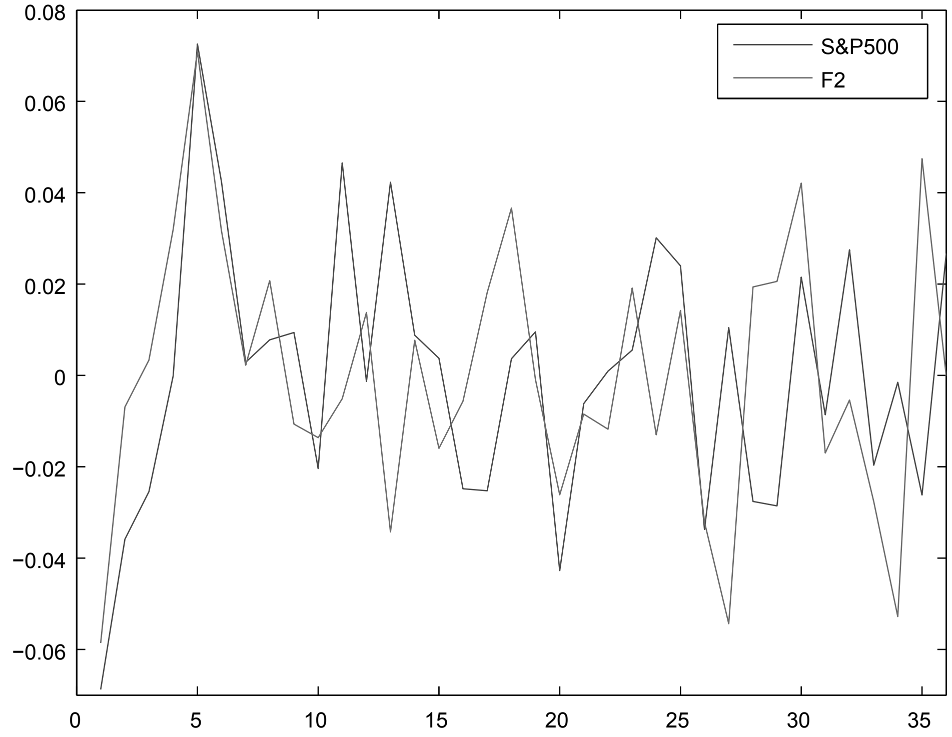


Figure 3: The second estimated latent factor F2 and the S&P 500 index returns from December 2002 to December 2005.



IV.3. ECONOMIC INTERPRETATION OF COMMON LATENT FACTORS

Once the latent factors and their number estimated, we implement tests proposed by Bai and Ng (2006) in order to match observed risk factors with estimated latent variables. We considered 2 sets of observed factors.

$i)$ The first set, called buy-and-hold risk factors, consists of several indexes: $1)$ S&P 500: the S&P 500 total return; the factors of Fama and French (1993) provided by the website of Kenneth French[13]: $2)$ SMB (small minus big): the spread between small and big capitalizations; $3)$ HML (high minus low): the spread between high and low Book-to-Market stocks; $4)$ MOM (momentum): the short-term reversal factor of Carhart[14](1994); $5)$ $CREDIT$: the spread between the Moody's BAA Corporate Bond Index return and the US Government $10-year$ yield; 6) $BOND$: the return on the Moody's Bond Index Corporate AA; $7)$ $CMDTY$: the Goldman Sachs Commodity Index (GSCI) total return; $8)$ USD : the U.S. Dollar Index return.

$ii)$ The second set consists of Agarwal and Naik (2004) option-based risk factors[15] represented by at-the-money (ATM) and out-of-the-money (OTM) European call and put options on the S&P 500. As discussed by the authors, the process of buying an ATM call option on the S&P 500 index consists of purchasing, on the first trading day of each month, an ATM call option on the S&P 500 that expires in the next month and selling the call option bought in the first day of the previous month. This procedure provides time series of returns on buying an ATM call option on the S&P 500. Similar procedures are used to get time series of returns for ATM put option, as well as OTM call and put options on the S&P 500. The ATM call (put) options on the S&P 500 are denoted by SPC_a (SPP_a), and the OTM call (put) options are denoted by SPC_o (SPP_o).

The Agarwal-Naik factors are highly correlated both among each other and with the S&P 500 index. To avoid some important drawbacks due to factor collinearity, such as beta instability, only the option-based factor having the lowest value of $NS(j)$ criterion is included in the analysis. For each option-based factor j ($j=1,2,3,4$), we use the rolling window procedure that will be exposed in Subsection IV.4 to calculate $NS(j)$, and then we get its average value across time $\overline{NS}(j)$. The SPP_a is the one having the lowest $\overline{NS}(j)$ ($\overline{NS}(j)=0.35$). Thus, this factor is added to the set of buy-and-hold factors, ending up with 9 observed variables to be considered in our analysis: $m=9$.

We must choose, among the 9 candidates, the relevant observed factors, which are generated by (or are linear combinations of) the estimated latent factors. We turn our attention toward the $NS(j)$ criterion[16] to select the factors to be included in the model. Monte Carlo simulation results given in Table 5 of Appendix C suggest that the observed variables may have $NS(j)$ values up to 20. Note that these values are much lower that those of the irrelevant factors. We thus set the critical value of $NS(j)$ up to 20. The observed factors carrying a $NS(j)$ value below 20, for any given rolling period, are included in the analysis[17]. For instance, focusing on the S&P 500, we obtain NS values close to zero, which means that this particular factor plays a significant role in explaining equity hedge fund returns. At the end, this procedure yields the 72 by 9 selection matrix S which is reported in the Appendix E. Each of the 72 rows of this matrix corresponds to a particular rolling period. Each element S_{jt} ($j = 1, \dots, m$, $t = 1, \dots, 72$) equals to one if the j th factor satisfies the restriction imposed on the $NS(j)$ criterion for the t th rolling period, and zero otherwise.

The selection matrix S highlights two important results concerning the risk exposures of the equity hedge fund returns.

$i)$ The S&P 500 index, the Fama-French factors as well as the option-based factor SPP_a are always relevant.

$ii)$ Factors such as USD or $CMDTY$ seem to be irrelevant for most cases, while the $CREDIT$ factor becomes relevant only at the beginning of 2000.

IV.4. DYNAMIC LINEAR REGRESSION ANALYSIS AND REPLICATION

The two previous subsections help determine the risk dimension and identify the observed factors that are generated by (or are linear combinations of) the common latent factors estimated from the data. This subsection presents the linear regression analysis and the hedge fund replication methodology. Hedge fund replication is used as a criterion for assessing the quality of the dynamic factor-based approach developed in this paper, as well as the benefits of dynamic factor selection mechanism. Note that our approach is similar to that of Hasanhobvic and Lo (2007). However, while they select in advance a set of observed factors and leave it unchanged through time, we allow for time-varying risk profile by using the factor selection methodology described in Sections 2 and 3.

We use a 37-month rolling window to estimate risk exposures for each fund i ($i = 1, \dots, N$) and construct an out-of-sample replicating portfolio. The first 36 months allow to $i)$ estimate the risk dimension h_t (*i.e.*, the number of the selected observed variables for a given rolling window); $ii)$ select the relevant observed factors $(G_1, \dots, G_{h_t})$ and $iii)$ perform the linear regressions given below:

$$R_{i,t-k} = \beta_{i,t}^{(1)} G_{1,t-k} + \dots + \beta_{i,t}^{(h_t)} G_{h_t,t-k} + \varepsilon_{i,t-k},$$

$$for\, k = 1, \dots, 36, \quad i = 1, \dots, N, \qquad (11)$$

$$subject \quad to \quad 1 = \sum_{j=1}^{h_t} \beta_{i,t}^{(j)}, \quad i = 1, \dots, N.$$

In this equation, $R_{i,t-k}$ denotes the return of fund i in $t-k$ and $\beta_{i,t}^{(j)}$ is the fund's i exposure to the j th factor. Beta coefficients are indexed by both i and t to reflect the fact that this process is repeated each month (using month $t-36$ to $t-1$ observations) for every fund i. To reflect that the number of observed factors considered for a given period is time-varying, h_t is also indexed by time. Thus, our approach is more general since account is taken not only of time-varying betas, but also of variability of hedge fund risk profile. In addition, selecting only the relevant factors eliminates noise due to model overfitting and improves the ability of the regression model to explain the observed data. Hasanhobvic and Lo (2007) approach can then be seen as a particular case of our procedure, when risk profile is constant in time.

Following Hasanhobvic and Lo (2007), we omit the intercept, which forces the least squares algorithm to use the factor means to fit the mean of the fund, which is an important feature of replicating hedge fund expected returns with factor risk premia. In addition, we constraint the sum of beta coefficients to be one in order to get a portfolio interpretation of the weights.

The estimated regression coefficients $\widehat{\beta}_{i,t}^{(h_t)}$ are then used as portfolio weights for the h_t observed factors. Hence, the replicated returns for the fund i are equivalent to the fitted values $\widehat{R}_{i,t}$ of the regression equation:

$$\widehat{R}_{i,t} = \widehat{\beta}_{i,t}^{(1)} G_{1,t} + \dots + \widehat{\beta}_{i,t}^{(h_t)} G_{h_t,t}. \qquad (12)$$

The results obtained using our dynamic approach are compared with those of a naive replication strategy, which consists in including in the regression analysis the whole set of the observed factors. In this case, all the elements of the selecting matrix S are set to one and the number of factors h is constant over time. In this case, equations (11) and (12) include all the observed factors for each rolling window.

IV.4.1 Empirical results

Applying the replication procedures exposed above, we get two equally-weighted replicating portfolios, called respectively the dynamic clone index and the naive clone index, by averaging the returns of individual fund clones at each date. Both replicating portfolios are compared to the equally-weighted equity hedge index built using the funds of our sample of data. Figure 4 plots the cumulative returns of the two clone indexes, the equally-weighted equity hedge index, as well as the S&P 500 for the whole replication period extending from January 2000 to December 2005 (72 months). The main summary statistics and the tracking errors of the replicating portfolios are reported in Table 1.

Our dynamic replicating approach outperforms the naive one consisting of a *static* and *ad hoc* factor selection procedure. This highlights the benefits of taking into account the time-varying risk profile when analyzing hedge fund returns. In particular, we measure the benefits of using our dynamic approach instead of the naive one by the reduction of both replication risk (the tracking error), as well as the distance between the target and the clone average returns. When we look at average returns, the dynamic clone index outperforms the naive clone index. The annualized average return for the dynamic clone is $8,06\%$ with a volatility of $8,95\%$ against an average return of $6,85\%$ for the naive clone with a volatility of $9,50\%$. The dynamic clone index has a lower tracking error ($1,23\%$) than the naive clone index ($1,55\%$). Thus, the dynamic replicating approach provides the best clone of the equity hedge index.

IV.4.2 Additional results

We first study the influence of the risk dimension estimating criterion in the quality of replication. The dynamic clone index reported in Figure 4 is obtained using Bai and Ng (2002) IC_2 criterion. In order to assess the benefits of using these criterion, we perform the dynamic approach using the other criteria (see Subsection II.2). The tracking errors are $1,28\%$ for the IC_1 criterion and $1,55\%$ for IC_3. Although the first alternative gives similar performance, it is not the case for the second one[18], which highlights the importance of a good criterion selection.

Second, we analyze the impact of dynamic factor selection procedure on beta turnover. In the previous paragraph, we show that accounting for time-variability of hedge fund risk profile improves the quality of hedge fund clone index. However, being time-dynamic may have a consequence in terms of beta turnover and other statistical properties of the different clones. At each replication date t ($t = 1,...,72$), we compute the turnover for a given individual hedge fund i ($i = 1,..., N$) as the sum of the absolute value of beta variations with respect to the previous period. Then, we take the average turnover across individual funds. Repeating this procedure for each rolling period, yields the time evolution of the average turnover, which is showed in Figure 5 for dynamic as well as naive approaches. The dynamic clone index outperformance as compared to the naive clone index is not necessarily due to higher fund turnover on average. Although turnover values are higher at individual fund level, they cancel each other when dealing with the index replication.

Third, we consider the replication quality at the individual fund level. Table 2 reports summary statistics for individual clones obtained by both dynamic and naive replication approach. Columns 2 to 5 give the means and the standard deviations of the annualized average returns as well as annualized return volatility. Columns 6 and 7 report the means and standard deviations of individual clone tracking errors. The results show that the dynamic strategy outperforms the naive one. The tracking errors of the 3 worst fund clones in terms of replication quality are $9,81\%$, $10,32\%$ and $11,51\%$ for the dynamic[19] approach, and $11,61\%$,

Figure 4: Cumulative returns of equally-weighted equity hedge fund portfolio, dynamic clone index, naive clone index, and the S&P 500 index

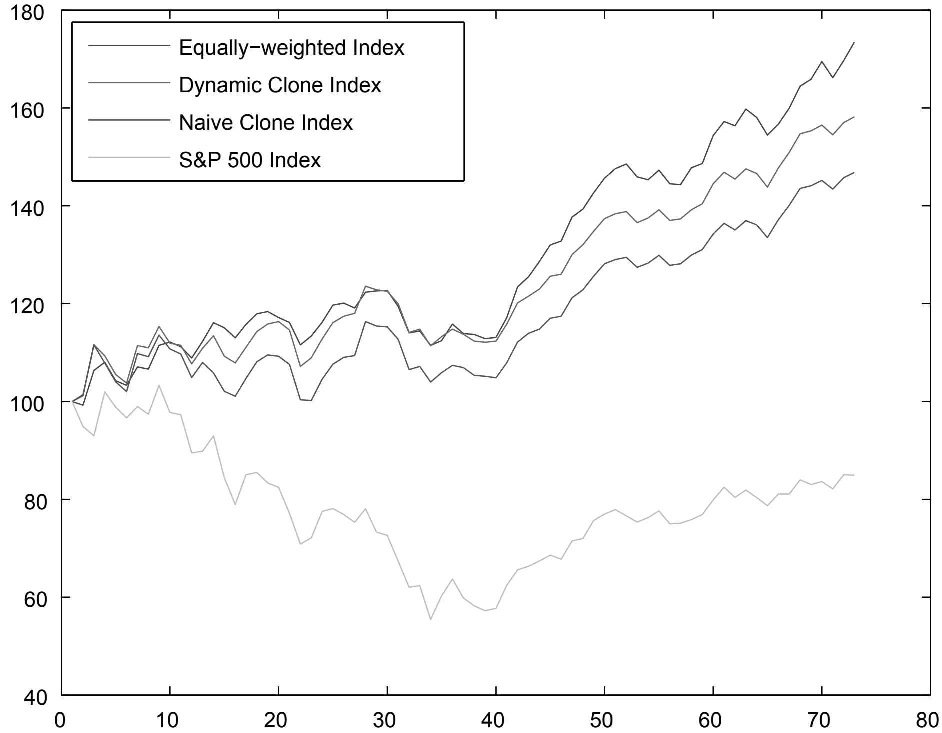
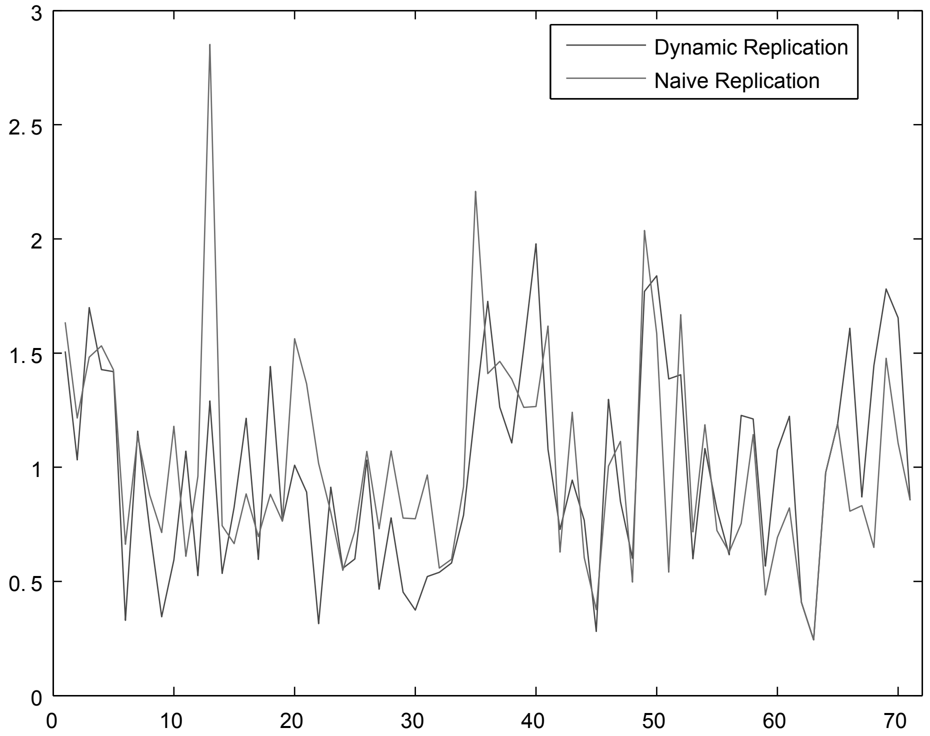


Table 1: Summary statistics for replication results (in percentage) using buy-and-hold and option-based factors

	Dynamic Clone Index	Naive Clone Index	Equity Hedge Index	S&P 500 Index
Tracking Errors	1,23	1,55	–	–
Annualized Return	8,06	6,85	9,50	0,02
Annualized SD	8,95	9,50	7,70	15,23

Figure 5: Time evolution of average turnover across individual funds for dynamic and naive replication strategies



$13,65\%$ and $14,35\%$ for the naive[20] approach. On the other hand, the three best funds have tracking errors close to zero for both approaches. Hence, at the individual level, the dynamic clones behave better than the naive ones.

Finally, we study the difference between the two approaches in terms of equity exposure. Figure 6 shows the time evolution of the average market betas across individual funds. At each replication date t we estimate market factor loadings of individual funds and take their average. Repeating this procedure 72 times yields the average beta time dynamics. The market beta is (on average) more stable when using the dynamic approach, which can be explained by a lower selection error due to the allowance for dynamic risk profile.

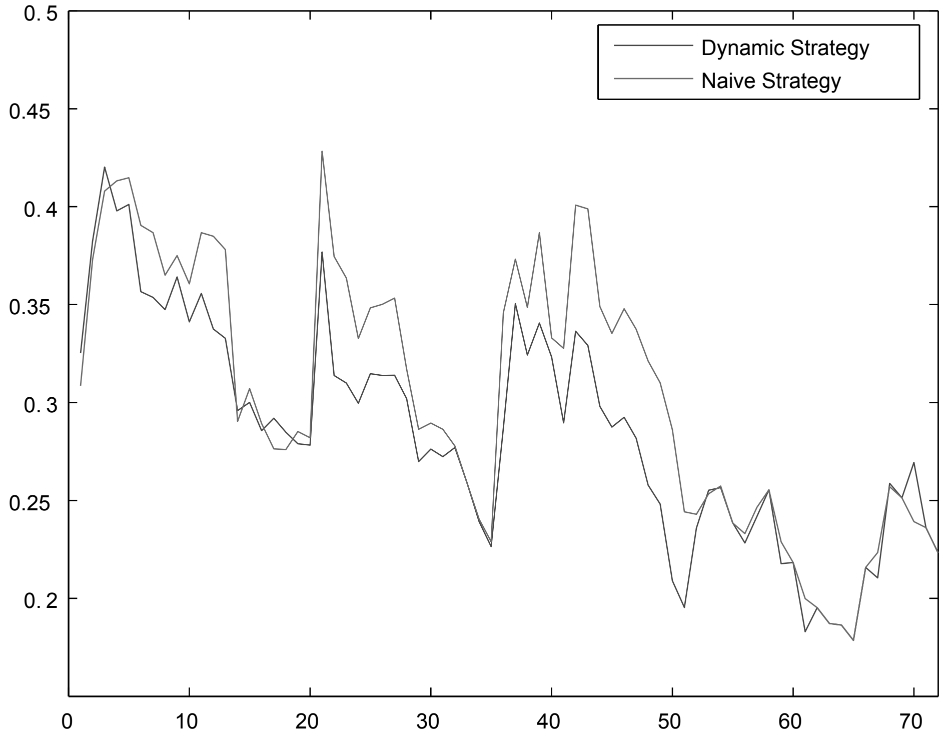
■ V. Concluding remarks

In this paper, we link a new market practice – hedge fund replication, to some useful and well known financial theory – factor modelling of equity returns. We get a deeper comprehension of the underlying factor structure that drives the covariations of equity hedge fund returns, using individual fund returns instead of index performances. Recent asymptotic theories for factor selection ensure good finite sample properties for large N and

Table 2: Summary statistics for individual funds (in percentage)

	Annualized Mean		Annualized SD		Tracking Error	
Individual Funds	11,43	6,90	13,01	7,27		
Dynamic Clones	8,77	6,61	9,12	6,53	2,91	1,74
Naive Clones	8,15	6,97	9,96	7,33	3,20	1,98

Figure 6: Time evolution of average market betas across individual funds for dynamic and naive procedures



moderately large T, which is clearly more in line with the dynamic factor selection objective we aim at.

This approach allows us to obtain several empirical results. First, equity hedge funds belonging to the HFR database exhibit a simple 2 - 3 factor structure. While the first factor behaves closely with the equity market index, the second one is often more difficult to understand and illustrates the style rotation employed by hedge fund managers. Second, the economic interpretation of the risk factors allows building replicating portfolios as they are proposed by practitioners. Our results highlight the interest of taking into account time-varying risk profile in the replication procedure at aggregated level but also at the individual one. The dynamic approach outperforms the naive one, which consists of a *static* and *ad hoc* factor selection procedure. ■

[1] As proxied by the difference between the Wilshire Small Cap 1750 and the Wilshire Large Cap 750 index.

[2] Note that several recent studies have challenged the absence of correlation of hedge fund returns with market indexes, arguing that the standard methods of assessing their risks may be misleading. For example, Asness, Krail and Liew (2001) show that in several cases where hedge funds purport to be market neutral, including both contemporaneous and lagged market returns as regressors and summing the coefficients yield significantly higher market exposure.

[3] Note that an alternative approach to observed factor models consists in using option-based factors to capture the nonlinearity of hedge fund returns. For example, Asness, Fung and Hseih (2001) show that the returns from trend following strategies can be replicated by a dynamically managed option-based strategy known as "lookback option". However, Amin and Kat (2003) point out that option-based models are difficult to be implemented in practice.

[4] In other words, their factor selection mechanism consists in selecting a given set of observed factors and keeping it unchanged through the entire period.

[5] In this paper we call it: the naive factor selection mechanism.

[6] The restriction $r = 1$ is implicitly made when we restrict the analysis to a single index built on hedge fund returns. We use the cross-section dimension to estimate this number from the data.

[7] For example, when $N > T$, the rank of $\widehat{\Sigma}$ is no more than T, whereas the rank of Σ can always be N.

[8] Bai and Ng (2002) also propose another set of criteria called Panel Criteria, PC, which present two main drawbacks as compared to IC criteria: $i)$ the PC criteria depend on $\hat{\sigma}^2 = V(k_{max}, \tilde{F}^{k_{max}})$ and, thus, on the arbitrary choice of k_{max} which is set to 8; $ii)$ Monte Carlo simulations performed by Bai and Ng (2002) show that, although asymptotically equivalent, IC criteria outperform PC criteria in terms of finite sample properties. For these reasons, only IC criteria are considered in this study.

[9] The idea to use statistic factors to price asset returns was first introduced by the arbitrage pricing theory (APT) developed by Roll (1976) and Ross (1980).

[10] The rate of convergence and the limiting distributions for the estimated factors, factor loadings and common components, estimated by the principal component method (PCA) was developed by Bai Ng (2003).

[11] As discussed in the previous section, F and r can consistently be estimated using asymptotic principal component computations when both sample dimensions are large.

[12] For each rolling window, we drop the funds that do not have full data for the given period. This allows us to count progressively for the new funds that enter in the database. The size of our sample varies from 97 for the first rolling window to 388 for the last one.

[13] The authors use data belonging to Center for Research in Security Prices (CRSP) of the University of Chicago.

[14] Which is also extracted from Kenneth French's website.

[15] We thank Vikas Agarwal for providing the data.

[16] We also considered the R^2 statistics and obtained similar results. The test results for each of the 72 rolling windows are available upon request.

[17] We try different critical values for $NS(j)$ and obtain similar results. In particular, we allow for $NS(j)$ critical value to vary from 10 to 20. The results reported in the next subsection consider a critical value equal to 15.

[18] We also perform the dynamic replicating approach using the PC_1, PC_2 and PC_3 criteria not reported here. The tracking errors are 1,52%, 1,52% and 1,53%, respectively.

[19] The tracking errors for the naive counterparts of these three worst dynamic clones are 13,65%, 10,97% and 11,67%, respectively.

[20] The tracking errors for the dynamic counterparts of these three worst naive clones are 9,24%, 9,81% and 9,51%, respectively.

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Appendices. A. Bai and Ng (2002) assumptions for approximate factor structure

To allow for some cross and serial correlation as well as heteroskedasticity in the idiosyncratic components, Bai and Ng (2002) make the following set of assumptions:

Assumptions

Time and cross-section dependence and heteroskedasticity: There exists a positive constant $M < \infty$, such that for all N and T,

1. $E(e_{it}) = 0$, $E\,|\,e_{it}\,|^8 \leq M$;

2. $E(e_s^{'}e_t^{'}\,/\,N) = E(N^{-1}\sum_{i=1}^{N} e_{is}e_{it}) = \gamma_N(s,t)$, $|\,\gamma_N(s,s)\,| \leq M$ for all s,

and $T^{-1}\sum_{s=1}^{T}\sum_{t=1}^{T}|\,\gamma_N(s,t)\,| \leq M$;

3. $E(e_{it}e_{jt}) = \tau_{ij,t}$ with $|\,\tau_{ij,t}\,| \leq |\,\tau_{ij}\,|$ for some τ_{ij} and for all t;

in addition, $N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}|\,\tau_{ij}\,| \leq M$;

4. $E(e_{it}e_{js}) = \tau_{ij,ts}$ and $(NT)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{t=1}^{T}\sum_{s=1}^{T}|\,\tau_{ij,ts}\,| \leq M$;

5. for every (t,s), $E\,|\,N^{-1/2}\sum_{i=1}^{N}[e_{is}e_{it} - E(e_{is}e_{it})]\,|^4 \leq M$.

Given Assumption 1, the remaining assumptions presented above are easily satisfied if the e_{it} are independent for all i and t. Assumptions 2 and 3 respectively allow for limited time-series and cross-section dependence in the idiosyncratic components. Heteroscedasticity in both dimensions is also allowed (Assumption 4). In addition the authors allow for some weak dependence between factors and the idiosyncratic errors, which is formalized by:

$$E\left(\frac{1}{N}\text{n}\,\frac{1}{\sqrt{T}}\sum_{t=1}^{T}F_t e_{it}\,\text{n}^{\,2}\right) \leq c.$$

B. Determining the number of latent factors: Monte Carlo simulations

Although asymptotically equivalent, the IC criteria do not have the same finite sample properties [see Bai and Ng (2002)]. We perform Monte Carlo simulations to assess the finite sample properties of the three information panel criteria given in (5). In particular, we are interested in cases of large N but moderately large T. As in Bai and Ng (2002), we simulate data from the following model[1]:

$$X_{it} = \sum_{j=1}^{r} \lambda_{ij} F_{jt} + \sqrt{\theta} e_{it} = c_{it} + \sqrt{\theta} e_{it}, \qquad (13)$$

where the idiosyncratic errors are generated by the following equation in order to allow for serial and cross-section correlation:

$$e_{it} = \rho e_{it-1} + \nu_{it} + \sum_{j \neq 0, j \neq -J}^{J} \beta \nu_{i-jt}. \qquad (14)$$

In this equation, ρ and β represent serial and cross-section correlation parameters, respectively, J is the number of the cross-correlated idiosyncratic components with θ being their variance.

The factors are $T \times r$ matrices of $N(0,1)$ variables and the factor loading are $N(0,1)$ variables. Hence, the common component of X_{it}, denoted by c_{it}, has variance r. Our model assumes that the idiosyncratic component has the same variance[2] as the common component (*i.e.*, $\theta = r$). We set $\rho = 0.50$, $\beta = 0.20$ and $J = \max[N/20, 10]$.

We consider thirteen configurations of the data. The first five simulate plausible asset pricing applications with two years of monthly data $(T = 24)$ for 100 to 300 series of asset returns. We then increase T to 36 months. The last three configurations are more general and are used to recall the results obtained by Bai and Ng (2002). For each data configuration, we use the procedure exposed in section 2 in order to estimate the number of factors $\widehat{r}$ and repeat the exercice 1000 times. Table 3 shows the test results, averaged across 1000 simulations, for $r = 2$ (columns 3 to 5) and $r = 3$ (columns 6 to 8). Three main remarks can be drawn.

$i)$ The IC_2 criteria outperforms (on average) IC_1 and IC_3;

$ii)$ For small T ($T = 24$), the three criteria lose (on average) their precision, even when N is large. For $T = 36$, they do better in inferring the number of common factors used to generate the data;

$iii)$ When T and N are both small, the criteria do not perform efficiently in inferring the appropriate number of factors. For example, for $N = 30$ and $T = 40$ both sets of criteria overestimate r.

Table 3: Simulation results for *IC* criteria.

		$r=2$			$r=3$		
N	T	IC_1	IC_2	IC_3	IC_1	IC_2	IC_3
300	24	2,08	2,02	2,3	2,79	2,75	2,89
260	24	2,59	2,31	3,56	2,82	2,75	2,88
200	24	2,24	2,05	2,92	2,63	2,48	2,94
150	24	3	2,48	4,68	2,86	2,52	3,81
100	24	4,26	3,42	6,2	3,56	2,8	5,55
300	36	2	2	2,02	3,26	3,1	3,97
260	36	2	2	2,01	3,13	3	3,74
200	36	2,6	2,29	4,19	2,98	2,89	3,45
150	36	2,35	2,15	3,76	2,99	2,96	3,16
100	36	3,33	2,66	5,31	4,1	3,32	6,34
200	60	2,27	2,07	4,52	3,26	3,06	5,48
200	100	3,44	2,39	7,84	4,4	3,39	7,98
30	40	1	1	1	1	1	1

[1] All computations are performed using Matlab. The programs used for Monte Carlo simulations and test statistic computations are available upon request.

[2] Bai and Ng (2002) also performed simulations allowing for the variance of the idiosyncratic component to be larger than that of the common component and yield similar results for the finite sample properties of their criteria.

C. Matching the latent factors with the observed variables: Monte Carlo simulations

We perform Monte Carlo simulations in order to asses the finite sample properties of the asymptotic tests $NS(j)$ and $R^2(j)$ using the same data configurations as in appendix B $(T=24,\quad 36$ and $N=100,\quad 150,\quad 200,\quad 250,\quad 300)$. We assume $F_{kt}\sim N(0,1)$, $k=1,\ldots,r$ and $e_{it}\sim N(0,\sigma_e^2(i))$, where e_{it} is uncorrelated with e_{jt} for $i\neq j,\quad i,\quad j=1,\quad \ldots,\quad N$. The factor loadings are standard normal, *i.e.* $\lambda_{ij}\sim N(0,1),\ j=1,\ldots,r,\ i=1,\ldots,N$. The data are generated as $X_{it}=\lambda_{it}'F_t+e_{it}$. We assume that there are $r=2$ factors and that this is known. The data are standardized to have mean zero and unit variance prior to the estimation of the factors by the method of asymptotic principal components. The observed factors are generated as $G_{jt}=\delta_j'F_t+\varepsilon_{jt}$ where δ_j' is a $r\times 1$ vector of weights, and $\varepsilon_{jt}\sim\sigma_\varepsilon(j)N(0,var(\delta_j'F_t))$. As in Bai and Ng (2006), we test $m=7$ observed variables parameterized as given in table 4:

Table 4: Parameters for G_{jt} simulation.

J	1	2	3	4	5	6	7
δ_{j1}	1	1	1	1	1	1	0
δ_{j2}	1	0	0	1	0	1	0
σ_ε	0	0	0.2	0.2	2	2	1

The first two factors, G_{1t} and G_{2t}, are exact factors since $\sigma_\varepsilon=0$. Factors three to six are linear combinations of the two latent factors but are contaminated by errors. The variance of this error is small relative to the variations of the latent factors for G_{3t} and G_{4t}, but is large for G_{5t} and G_{6t}. Finally, G_{7t} is an irrelevant factor as it is simply a random variable $N(0,1)$.

The Monte Carlo simulation results are reported in Table 5. The test statistics are averaged over 1000 simulations. The $NS(j)$ and $R^2(j)$ statistics reinforce the previous result. When the observed factors are contaminated by errors, Table 5 shows that the higher the variance of this error, the worse is the efficiency of the tests considered here. Finally, the test precision is higher for $T=36$ than for $T=24$.

Table 5: Simulation results: Matching the observed variables to latent factors

N	T		$NS(j)$	$R^2(j)$	N	T		$NS(j)$	$R^2(j)$
300	36	G_1	0,009	0,991	300	24	G_1	0,009	0,991
300	36	G_2	0,009	0,991	300	24	G_2	0,009	0,991
300	36	G_3	0,085	0,923	300	24	G_3	0,085	0,923
300	36	G_4	0,046	0,956	300	24	G_4	0,046	0,956
300	36	G_5	20,338	0,163	300	24	G_5	33,421	0,188
300	36	G_6	5,697	0,246	300	24	G_6	8,64	0,262
300	36	G_7	567,107	0,057	300	24	G_7	46,731	0,087

C. (suite)

Table 5 (suite)

N	T		$NS(j)$	$R^2(j)$	N	T		$NS(j)$	$R^2(j)$
250	36	G_1	0,011	0,99	250	24	G_1	0,012	0,989
250	36	G_2	0,011	0,989	250	24	G_2	0,011	0,989
250	36	G_3	0,087	0,92	250	24	G_3	0,085	0,922
250	36	G_4	0,049	0,954	250	24	G_4	0,049	0,954
250	36	G_5	20,927	0,159	250	24	G_5	18,144	0,187
250	36	G_6	6,616	0,243	250	24	G_6	29,687	0,267
250	36	G_7	128,404	0,059	250	24	G_7	60,498	0,086
200	36	G_1	0,013	0,987	200	24	G_1	0,013	0,987
200	36	G_2	0,013	0,987	200	24	G_2	0,014	0,986
200	36	G_3	0,089	0,919	200	24	G_3	0,089	0,919
200	36	G_4	0,052	0,951	200	24	G_4	0,052	0,951
200	36	G_5	18,9	0,163	200	24	G_5	28,367	0,191
200	36	G_6	5,554	0,24	200	24	G_6	6,084	0,269
200	36	G_7	127,262	0,055	200	24	G_7	80,598	0,086
150	36	G_1	0,018	0,982	150	24	G_1	0,019	0,981
150	36	G_2	0,017	0,983	150	24	G_2	0,02	0,981
150	36	G_3	0,097	0,913	150	24	G_3	0,093	0,916
150	36	G_4	0,057	0,946	150	24	G_4	0,057	0,947
150	36	G_5	19,742	0,158	150	24	G_5	23,36	0,192
150	36	G_6	5,856	0,245	150	24	G_6	5,701	0,266
150	36	G_7	97,851	0,059	150	24	G_7	75,996	0,088
100	36	G_1	0,027	0,974	100	24	G_1	0,028	0,973
100	36	G_2	0,027	0,974	100	24	G_2	0,029	0,972
100	36	G_3	0,105	0,905	100	24	G_3	0,106	0,905
100	36	G_4	0,066	0,938	100	24	G_4	0,067	0,938
100	36	G_5	18,603	0,154	100	24	G_5	19,214	0,184
100	36	G_6	21,352	0,234	100	24	G_6	7,83	0,26
100	36	G_7	102,409	0,057	100	24	G_7	83,84	0,085

D. Data description

Table 6: Fund repartition by strategy for the HFR Database in December 2005

	Strategy	Fund Number
1	Convertible Arbitrage	109
2	Distressed Securities	127
3	Emerging Markets	269
4	Equity Hedge	1232
5	Equity Market Neutral	282
6	Equity Non-Hedge	146
7	Event-Driven	225
8	Fixed Income	310
9	Foreign Exchange	68
10	Fund of Funds	2011
11	Macro	277
12	Managed Futures	337
13	Market Timing	25
14	Merger Arbitrage	46
15	Relative Value Arbitrage	268
16	Sector	279
17	Short Selling	23
	Total	**6034**

E. Selection matrix for dynamic replication strategy

Table 7: Selection matrix (1)

	S&P 500	SMB	HML	MOM	CREDIT	BOND	CMDTY	USD	SPP_a
1	1	1	1	1	1	0	1	0	1
2	1	1	1	1	1	0	1	0	1
3	1	1	1	1	1	0	1	0	1
4	1	1	1	1	0	0	1	0	1
5	1	1	1	1	0	0	1	0	1
6	1	1	1	1	0	0	0	0	1
7	1	1	1	1	0	0	0	0	1
8	1	1	1	1	0	0	1	0	1
9	1	1	1	1	0	0	1	0	1
10	1	1	1	1	0	0	1	0	1
11	1	1	1	1	0	0	1	0	1
12	1	1	1	1	0	0	0	0	1
13	1	1	1	1	0	0	0	0	1
14	1	1	1	1	0	0	0	0	1
15	1	1	1	1	0	0	0	0	1
16	1	1	1	1	0	0	1	0	1
17	1	1	1	1	1	0	1	0	1
18	1	1	1	1	1	0	1	0	1
19	1	1	1	1	1	0	0	1	1
20	1	1	1	1	1	0	0	0	1
21	1	1	1	1	1	0	0	0	1
22	1	1	1	1	1	0	0	0	1
23	1	1	1	1	1	0	0	0	1
24	1	1	1	1	1	0	1	0	1
25	1	1	1	1	1	0	1	0	1
26	1	1	1	1	1	0	1	0	1
27	1	1	1	1	1	0	1	0	1
28	1	1	1	1	1	0	1	0	1
29	1	1	1	1	1	0	1	0	1
30	1	1	1	1	1	0	1	0	1
31	1	1	1	1	1	0	1	0	1
32	1	1	1	1	1	0	1	0	1
33	1	1	1	1	1	0	1	0	1
34	1	1	1	1	1	0	1	0	1
35	1	1	1	1	1	0	1	0	1
36	1	1	1	1	1	0	1	0	1

E. (suite)

Table 8: Selection matrix (2)

	S&P 500	SMB	HML	MOM	CREDIT	BOND	CMDTY	USD	SPP$_a$
37	1	1	1	1	1	0	1	0	1
38	1	1	1	1	1	0	1	0	1
39	1	1	1	1	1	0	0	0	1
40	1	1	1	1	1	1	0	1	1
41	1	1	1	1	1	0	1	0	1
42	1	1	1	1	1	0	1	0	1
43	1	1	1	1	1	0	1	0	1
44	1	1	1	1	1	0	1	0	1
45	1	1	1	1	1	0	0	0	1
46	1	1	1	1	1	0	0	0	1
47	1	1	1	1	1	1	0	0	1
48	1	1	1	1	1	1	0	0	1
49	1	1	1	1	1	1	0	0	1
50	1	1	1	1	1	1	0	0	1
51	1	1	1	1	1	0	0	0	1
52	1	1	1	1	1	0	0	1	1
53	1	1	1	1	1	0	1	1	1
54	1	1	1	1	1	0	1	1	1
55	1	1	1	1	1	0	1	1	1
56	1	1	1	1	1	0	1	1	1
57	1	1	1	1	1	0	1	1	1
58	1	1	1	1	1	1	1	1	1
59	1	1	1	1	1	1	1	0	1
60	1	1	1	1	1	1	1	1	1
61	1	1	1	1	1	1	0	0	1
62	1	1	1	1	1	1	1	1	1
63	1	1	1	1	1	1	1	1	1
64	1	1	1	1	1	1	1	1	1
65	1	1	1	1	1	1	1	1	1
66	1	1	1	1	1	1	1	1	1
67	1	1	1	1	1	0	1	1	1
68	1	1	1	1	1	0	1	1	1
69	1	1	1	1	1	1	1	1	1
70	1	1	1	1	1	0	1	0	1
71	1	1	1	1	1	1	1	1	1
72	1	1	1	1	1	1	1	1	1